A Programmer Plays Darts

Doug Osborne (DougO1081@gmail.com)

Note: These are mostly unedited notes. For a summary of this work, see: https://doug-osborne.com/darts/

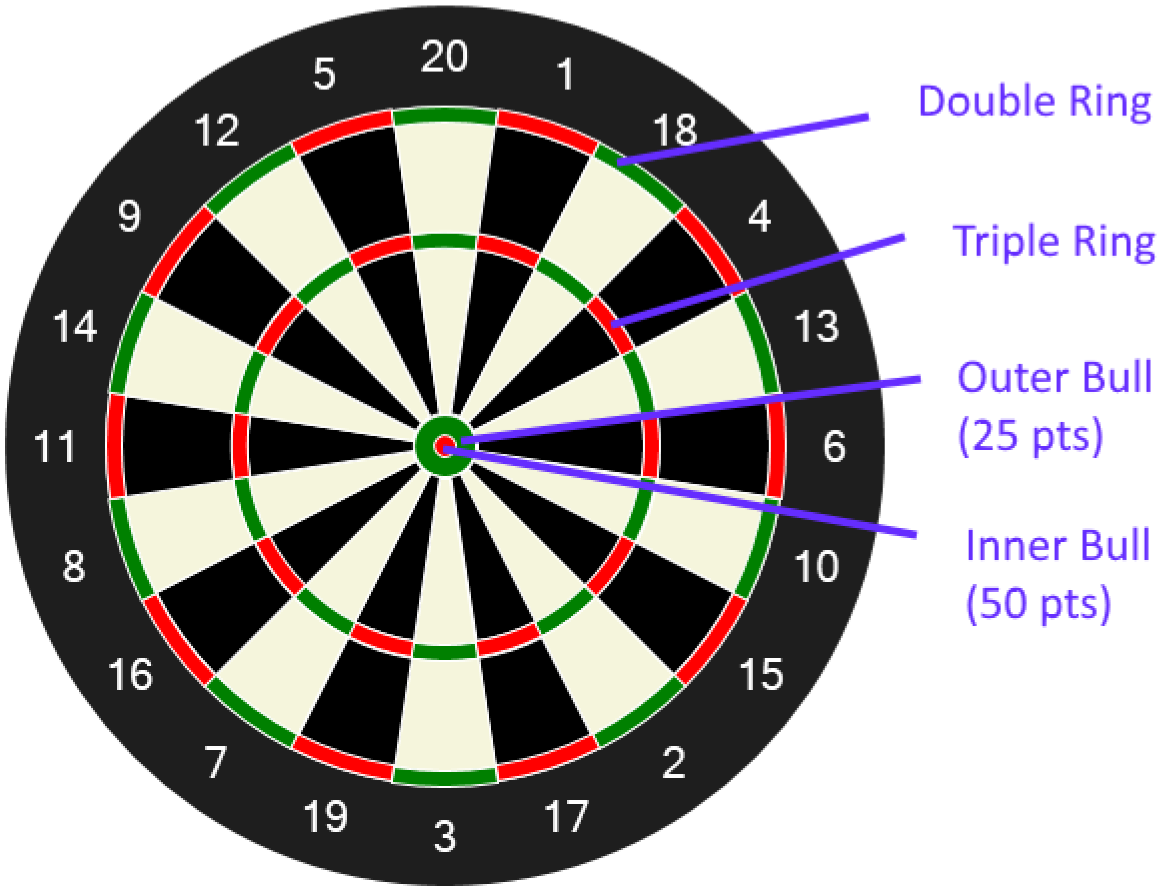
Preface

When I was about 12 years old my father purchased an electronic dartboard. This led to the creation of fantasy dart leagues - where we would throw darts for fictional “players” (with rather unusual names) on fictional teams, with each season culminating in a playoff to determine a champion.

This was my introduction to programming - all of the record keeping was automated, and in some later seasons there were even computer simulated players/teams. It also sparked my interest in the game of Darts, and specifically the problem of target optimization. Almost 15 years after our last season, I found myself with some free time and decided to tackle this problem.

In Part 1 I will take a purely mathematical approach to the problem, while in Part 2 I will put these results to the test by throwing a lot of darts.

The Dartboard



Part 1 - A Program Plays Darts

Question: If I want to maximize my chances of winning a game of darts, where should I aim?

Short Answer: It depends on the rules of the game, your skill level and your opponents skill level.

In this part I will first answer the simpler question of score maximization, while introducing the protagonist (Pigfoote) for the rest of Part 1.

I will then solve the win maximization problem for the Hi-Score variant of darts for Pigfoote before attempting to solve the more challenging variant (501) played in competitive darts today.

For the 501 variant, I will first build two strategy tables for Pigfoote based on different goals. I will then compare Pigfoote to two other players of similar skill but different throwing patterns.

Next I will consider 5 scenarios in a hypothetical match between Pigfoote and a [real darts player](#) where Pigfoote's optimal target may follow one of these two strategies, fall somewhere in between the two, or be somewhere else.

I will then demonstrate how to build a full win optimization strategy for Pigfoote that consists of targets for over 5 million specific situations. Finally, I will examine potential issues with this solution if applied to a real darts player, leading into Part 2.

When I set out to answer the Question, I intentionally avoided searching the internet because I wanted to find the solution myself.

Not surprisingly, once I did, it turned out that I was not the first one to tackle the problem (or at least, a similar one).

Maximizing Expected Score

Of the two papers and one website referenced on this page, each attempts to answer this question.

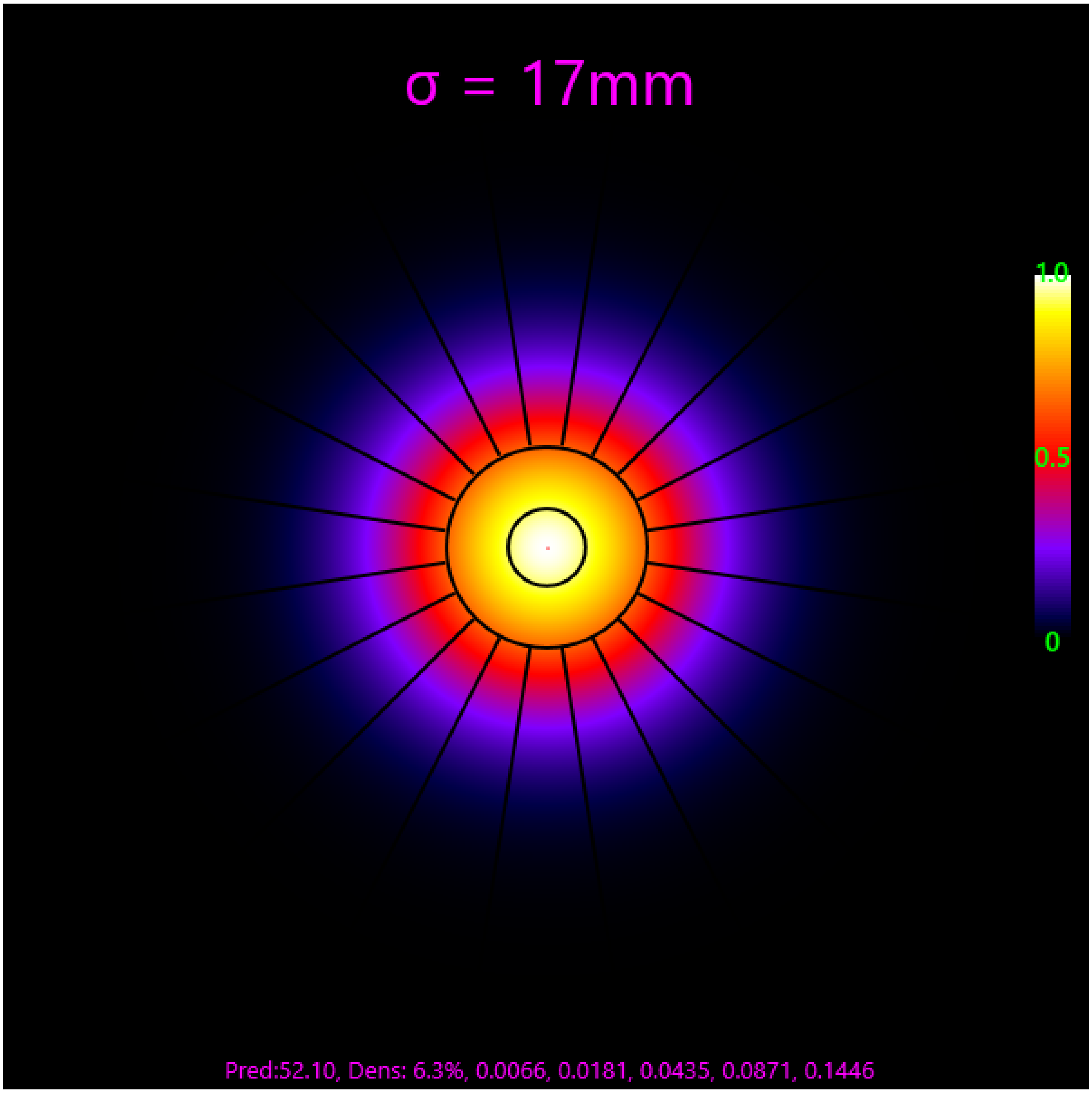
All three agree on certain basic principals, which I also arrived at prior to checking my work:

- The key is to create a probabilistic density function (pdf) centered around a target point. In layman's terms, this means that we can assign a relative likelihood of a dart landing at each point on the dartboard based on the pdf.
- With this function, we can sweep the dartboard and calculate the expected score for each target point.
- The most logical pdf to approximate a human's throwing is called a bivariate normal distribution (also called a Gaussian distribution).

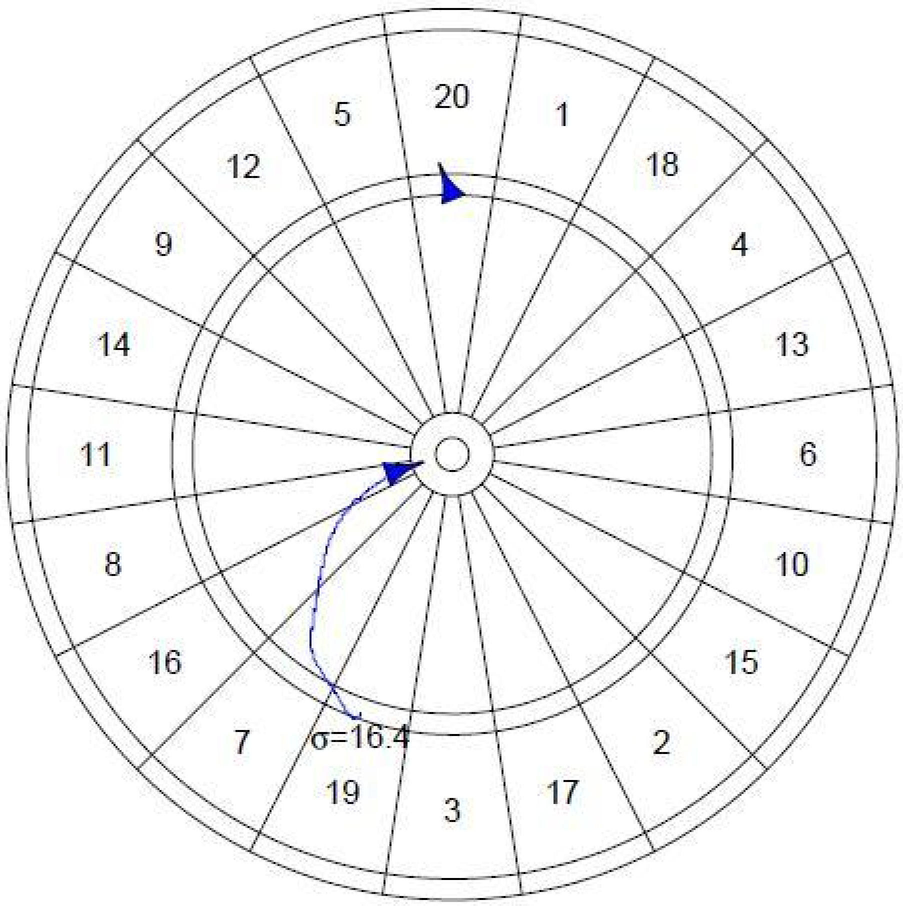
The term **bivariate** normal distribution is a fancy way of saying that the horizontal and vertical distance of a player's throws from their target follow a standard normal distribution (the 68/95/99.7 rule).

This particular pdf requires at least 1 variable corresponding to the throwers skill: their standard deviation (σ).

Here is what it looks like with an σ of 17 mm (1/10th the radius of a dartboard's scoring area):



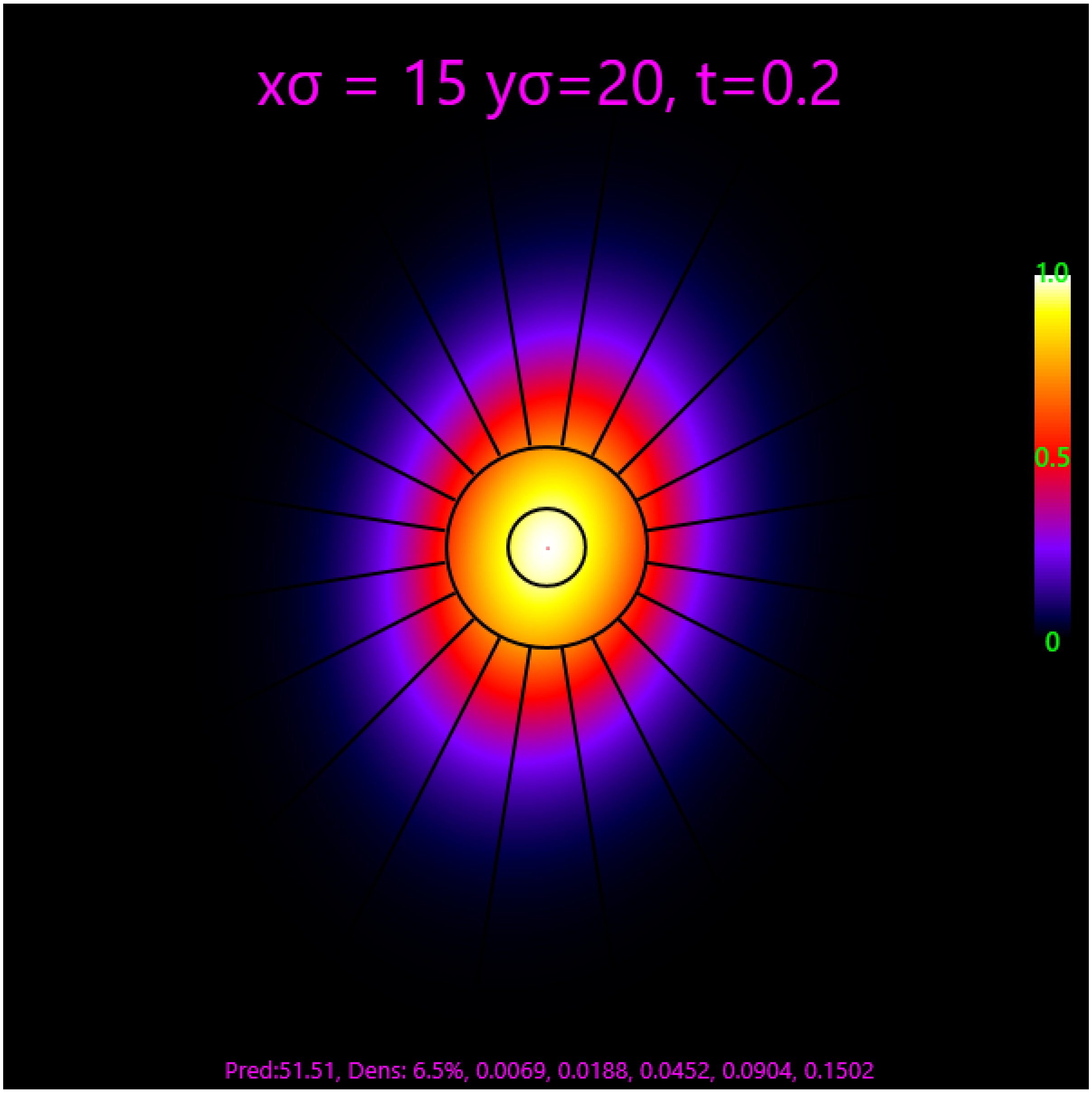
The website a geek plays darts does an excellent job explaining how to use this to find the optimal scoring target. The optimal scoring target with this model is the t20 when σ is under 16.4mm, then shifts to the t19 and curls around to the 7, 16 and finally to the left side of the bullseye.



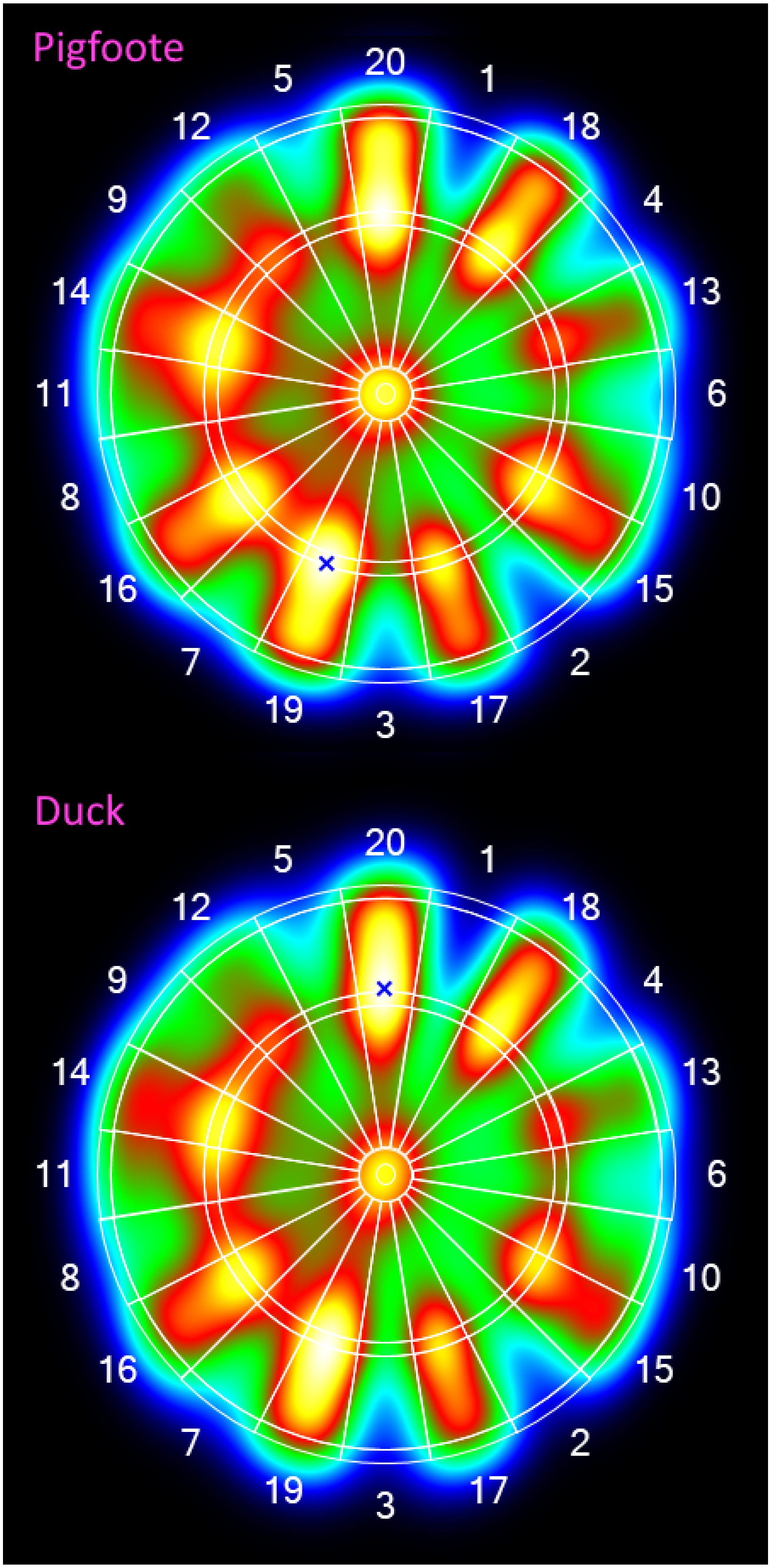
I will refer to the player with an σ of 17mm as "Pigfoote" - note that Pigfoote's σ is just above the threshold where the optimal scoring target switches from the T20 to the T19, where he should average about 19.7 points per throw.

The research paper A statistician plays darts expands on the basic pdf with a single σ. It considers separate x and y coordinate σ values as well as a tilt (a correlation between the x and y coordinates) and a skew (uneven distribution on right/left of center and/or above/below center).

Here is an example for a player I will call "Duck", with a 15 mm xσ, a 20 mm yσ and a 20% bottom left to top right tilt:



The following heat maps for Pigfoote and Duck show the relative value of each target along with the optimal scoring targets for each player:


Pigfoote
20
1
18
4
13
6
10
15
2
17
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16
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11
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9
12
5
Duck
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5

Both the geek and statistician playing darts stopped at this point - which does not quite answer my question.

Maximizing Expected Winning Percentage (Hi-Score)

For this problem, I will consider two variants of darts.

First is the simple “Hi-Score” variant that my father and I played in our fantasy dart leagues:

Each team has 3 turns with 3 throws per turn, and the team with the highest score wins. The teams alternate turns, with the visiting team throwing first. In our leagues, each turn is taken by a different “player”, though the only real players were myself, my father, and in some later seasons computer simulated players.

The benefit of this format is that we could play a lot of games quickly, with statistics kept for players that allowed for such things as all-star games, mvps, and even a fictional hall of fame.

To solve the target problem for this variant of darts, we first convert each score on the board to a winning percentage. Then we use the same sweep method as before to calculate the expected win percentage for each target and score differential. The format lends itself particularly well to this conversion by starting with the last throw and working backwards.

With one throw remaining, the values for each score are either 1 (a win), 0 (a loss), or 0.5 (a tie). Margin of victory does not matter.

If the home team is ahead already, every possible score is a 1, while if they are behind by 61 or more points, every score is a 0.

For each score differential where the outcome isn't certain (home team trailing by 0 to 60 points), we find the optimal target and the expected win percentage for each target.

Below is Pigfoote's strategy table for the last throw of the game. Pigfoote is a fairly skilled player - he can hit a slice of the pie over ¾ of the time (indicated by the 76.7% win rate trailing by 19), for example.

Deficit	Target	Win %
0-2	IS - 11: H: 0.50 V: 0.76*	100.00%
3	IS - 11: H: 0.58 V: 0.81*	100.00%
4	IS - 11: H: 0.62 V: 0.81*	100.00%
5	IS - 8: H: 0.84 V: 0.85*	100.00%
6	IS - 8: H: 0.76 V: 0.86*	100.00%
7	IS - 11: H: 0.94 V: 0.95	100.00%
8	T - 14: H: 0.94 V: 0.82	99.97%
9	OS - 14: H: 0.07 V: 0.29	98.12%
10	OS - 14: H: 0.01 V: 0.34	97.50%
11	OS - 14: H: 0.38 V: 0.45	84.29%
12-13	OS - 14: H: 0.49 V: 0.54	78.20%
14	OS - 16: H: 0.50 V: 0.53	77.61%
15	OS - 16: H: 0.52 V: 0.52	77.44%
16-18	OS - 19: H: 0.50 V: 0.52	77.12%
19	OS - 20: H: 0.50 V: 0.55	76.72%
20	OS - 20: H: 0.50 V: 0.69	41.80%
21-24	DB - 6: H: 0.50 V: 0.00	35.53%
25	DB - 6: H: 0.50 V: 0.00	21.19%
26	T - 9: H: 0.51 V: 0.33	18.84%
27	T - 14: H: 0.10 V: 0.35	17.95%
28-29	T - 10: H: 0.99 V: 0.37	17.65%
30-32	T - 14: H: 0.00 V: 0.41	17.58%
33-39	D - 20: H: 0.50 V: 0.38	16.32%
40-44	T - 17: H: 0.50 V: 0.51	12.40%

45-47	T - 19: H: 0.50 V: 0.51	12.40%
48	T - 19: H: 0.48 V: 0.47	12.38%
49-50	T - 17: H: 0.52 V: 0.47	12.36%
51	T - 19: H: 0.48 V: 0.47	12.33%
52-53	T - 18: H: 0.49 V: 0.43	12.32%
54-56	T - 19: H: 0.50 V: 0.51	12.31%
57-59	T - 20: H: 0.50 V: 0.50	12.15%
60	T - 20: H: 0.50 V: 0.50	6.07%

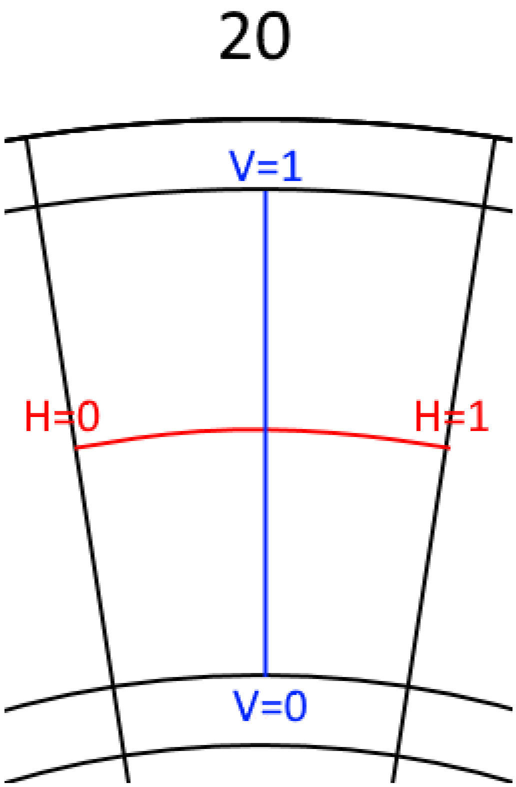
*- With a deficit of 6 or less points, there are many targets that produce a 100% win probability according to this model.

Target Notation: *Each number slice is divided into 7 sectors - from inner to outer: double bull (DB), single bull (SB), inner single (IS), triple (T), outer single (OS), double (D), miss (M).*

"H" measures the angle of the target clockwise - an H of 0 shares the border with the adjacent number counter clockwise (e.g, for 14 this is the 11) to the target, and an H of 1 shares a border with the adjacent number clockwise to the target (e.g, for 14 this is the 9).

"V" measures how far out from the center the point is relative to the sector. A V of 0 shares a border with the neighboring sector closer to the bullseye, and a V of 1 shares a border with the neighboring sector further away from the bullseye.

The following shows the H-V scale for the "Outer Single 20" zone. The central point of each sector has an H and V of 0.5. Note that H moves in a clockwise direction, and not right to left. A H of 0 for the 3 is on the right edge, not the left.



Note that any DB with a V of 0 is equivalent to the center of the double bull.

Key drop offs in win expectancy occur from deficits of 10 to 11 to 12 points (97.5-78.2%), from 19 to 20 to 21 points (76.7-35.5%), and from 24 to 25 points (35.5-21.2%), the sum differential of 5 points covers a full 72.8% change in win expectancy.

Trailing by exactly 10 points for example, the optimal target is between the 14 and 11 - this is the only place on the dartboard where there are two adjacent singles that will win the game. This explains the first large drop (13.3%) in win percentage from 10 to 11 points.

The win percentage values from this table are then used to assign win percentages to scores for the 2nd to last throw in the game. For example, since we know that a deficit of 19 with 1 throw remaining has an expected win % of 76.72%, any score that reduces the deficit to 19 on the 2nd to last throw is also worth 0.7672 wins.

Here is Pigfoote’s full 2nd to last throw chart:

Deficit	Target	Win %
0-2	IS - 11: H: 0.58 V: 0.76	100.00%
3-4	IS - 11: H: 0.62 V: 0.81	100.00%
5-6	IS - 8: H: 0.76 V: 0.86	100.00%
7	IS - 14: H: 0.00 V: 0.95	100.00%
8	T - 14: H: 0.99 V: 0.09	100.00%
9	IS - 11: H: 0.99 V: 0.98	100.00%
10	IS - 11: H: 1.00 V: 0.97	100.00%
11	IS - 14: H: 0.50 V: 0.94	100.00%
12-13	IS - 14: H: 0.47 V: 0.93	100.00%
14	IS - 16: H: 0.79 V: 0.96	100.00%
15	IS - 11: H: 0.94 V: 0.94	100.00%
16	IS - 14: H: 0.33 V: 0.98	100.00%
17	T - 14: H: 0.63 V: 0.56	99.99%
18	OS - 14: H: 0.03 V: 0.12	99.91%
19	OS - 14: H: 0.21 V: 0.20	99.80%
20	OS - 14: H: 0.15 V: 0.21	98.75%
21	OS - 14: H: 0.12 V: 0.20	98.37%
22	OS - 14: H: 0.44 V: 0.30	95.54%
23	OS - 16: H: 0.51 V: 0.32	94.49%
24	OS - 16: H: 0.51 V: 0.32	94.44%
25	OS - 16: H: 0.51 V: 0.23	93.15%
26	OS - 16: H: 0.52 V: 0.20	92.70%
27	OS - 19: H: 0.52 V: 0.47	85.87%
28	OS - 20: H: 0.45 V: 0.51	83.23%

29	OS - 20: H: 0.45 V: 0.51	81.87%
30	OS - 20: H: 0.48 V: 0.53	79.70%
31	OS - 14: H: 0.48 V: 0.04	71.13%
32	OS - 14: H: 0.51 V: 0.04	70.23%
33	OS - 14: H: 0.51 V: 0.04	69.98%
34	OS - 15: H: 0.43 V: 0.51	66.59%
35	OS - 19: H: 0.51 V: 0.56	64.86%
36-37	OS - 19: H: 0.51 V: 0.56	64.73%
38	OS - 19: H: 0.51 V: 0.56	64.44%
39	OS - 20: H: 0.50 V: 0.58	64.10%
40	DB - 6: H: 0.50 V: 0.00	43.32%
41	DB - 6: H: 0.50 V: 0.00	42.30%
42	DB - 20: H: 0.50 V: 0.16	41.56%
43	DB - 6: H: 0.50 V: 0.00	40.83%
44	DB - 20: H: 0.50 V: 0.16	39.99%
45-46	T - 14: H: 0.51 V: 0.39	31.24%
47	T - 14: H: 0.26 V: 0.36	29.76%
48	T - 14: H: 0.26 V: 0.36	29.61%
49-50	T - 14: H: 0.22 V: 0.45	29.31%
51	T - 14: H: 0.32 V: 0.42	28.15%
52	T - 14: H: 0.29 V: 0.39	27.80%
53	T - 15: H: 0.45 V: 0.46	26.05%
54	T - 15: H: 0.45 V: 0.46	25.82%
55	T - 16: H: 0.48 V: 0.50	25.57%
56-58	T - 19: H: 0.50 V: 0.51	25.49%

59	T - 20: H: 0.50 V: 0.50	25.15%
60	T - 19: H: 0.51 V: 0.39	23.33%
61-62	T - 19: H: 0.50 V: 0.51	23.20%
63	T - 19: H: 0.57 V: 0.47	22.44%
64-65	T - 19: H: 0.63 V: 0.56	21.79%
66	T - 19: H: 0.63 V: 0.56	21.56%
67	T - 19: H: 0.56 V: 0.59	20.41%
68	T - 20: H: 0.47 V: 0.63	19.28%
69-70	T - 20: H: 0.47 V: 0.63	19.05%
71	T - 20: H: 0.47 V: 0.63	17.36%
72	T - 19: H: 0.52 V: 0.67	16.62%
73	T - 20: H: 0.47 V: 0.63	16.58%
74-75	T - 19: H: 0.52 V: 0.67	16.57%
76	T - 19: H: 0.52 V: 0.67	16.41%
77-79	T - 20: H: 0.50 V: 0.75	15.95%
80	T - 20: H: 0.50 V: 0.63	8.36%
81-82	DB - 6: H: 0.50 V: 0.00	4.73%
83-84	DB - 6: H: 0.50 V: 0.00	4.60%
85	DB - 6: H: 0.50 V: 0.00	2.86%
86	T - 14: H: 0.65 V: 0.32	2.31%
87	T - 14: H: 0.10 V: 0.35	2.22%
88-89	T - 19: H: 0.50 V: 0.51	2.18%
90-92	T - 14: H: 0.00 V: 0.41	2.16%
93-96	T - 19: H: 0.50 V: 0.51	2.02%
97-99	T - 20: H: 0.50 V: 0.50	1.99%

100-107	T - 19: H: 0.50 V: 0.51	1.54%
108-111	T - 19: H: 0.48 V: 0.47	1.53%
112-113	T - 19: H: 0.50 V: 0.51	1.51%
114	T - 20: H: 0.50 V: 0.38	1.50%
115-116	T - 19: H: 0.50 V: 0.51	1.50%
117-119	T - 20: H: 0.50 V: 0.50	1.48%
120	T - 20: H: 0.50 V: 0.50	0.74%

This is where I found the results interesting - the major drop off points are somewhat less obvious and some of the targets take advantage of sequences of numbers that my father and I never really thought of.

The target of 16 trailing by 23 to 26 points is one example, which is based on the following:

- a 10 point or less deficit on the last throw is effectively a "win" for this player, and hitting a 16 achieves this goal.
- The scores to the left (7) and right (8) cross the key 20 point threshold.

Note the target is somewhat near the triple zone (v of 0.2 to 0.33), adding another chance to end the game immediately by hitting the T16, T7 or T8. The target is far superior to any other particularly at a deficit of 25-26 points - as the chances of losing nearly double with a 27 point deficit. My father and I undoubtedly lost a few games we could have won by not being aware of this (we would usually go for a 19, 20 or bull depending on which target we favored at the time).

Another interesting score is 40 - where this recommends Pigfoote goes for a win by aiming at the bull as opposed to a tie by shooting for two 20s.

The process can continue all the way to the first throw of the game, with the relevant score differentials tested. The maximum range is 540 points on the 10th throw of the game, covering a lead of 180 points to a deficit of 360 points for the home team. Beginning with the 4th to last throw of the game, we need to know both player's throwing distributions.

To show the relative frequency of targets, I simulated a million games between two teams of Pigfeet, where the targets were chosen according to the chart. When the outcome is certain, the target defaults to his optimal scoring target, the T19. The results can be grouped into two categories. Here are the first two rounds of the game:

Target	Away	Home
T/OS 19	69.61%	68.12%
T/OS 20	30.38%	29.52%
T/OS 14	0%	0.86%
OS 19	0%	0.85%
All Others	0.01%	0.65%

I classified any target within the triple sector or the outer single with a V < 0.1 as T/OS. Although Pigfoote's optimal scoring target is the T19, he scores very nearly as well aiming at the T20 which often compares favorably to the T19 in terms of individual scoring. Altogether different targets start occurring on the home team's 2nd turn.

My one takeaway from this table is that my father and I probably adopted a conservative strategy (aiming at the T14) somewhat more frequently than we should have.

Here is the table for the last turn of the game:

Target	Away	Home
T/OS 19	31.38%	11.97%*
T/OS 20	46.84%	11.27%
T/OS 14	8.76%	8.81%
OS 19	4.09%	4.17%
OS 20	3.95%	3.49%
Bull	1.11%	4.36%
OS 14	0.33%	4.45%
T/OS 16	2.72%	0.48%
D 20	0%	2.68%
OS 16	0.39%	1.95%

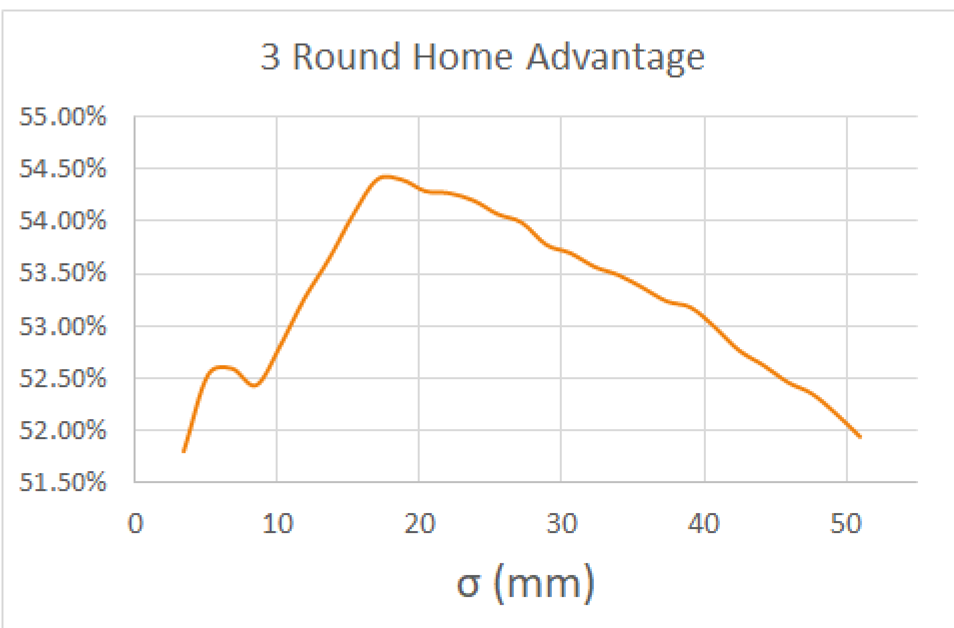
IS/T 14	0.01%	1.97%
IS/T 11	0.03%	1.32%
T/OS 17	0%	1.19%
T/OS 15	0.12%	1.03%
IS 8	0.04%	1.02%
All Others	0.23%	2.91%

*- I excluded the 37% of throws where the outcome was already decided and the target defaulted to the T19.

It should come as no surprise that the home team's strategy varies far more on the last turn of the game compared to the visitor's.

As a result, the home team has a considerable advantage and should expect to win about 54.4% of the time in this matchup (similar to the home field advantage in baseball).

This advantage varies depending on the skill level of the players, peaking right around Pigfoote's σ of 17mm:



For much stronger players than Pigfoote, the game is basically a triple 20 competition, while much weaker players aren't skilled enough to take full advantage of the extra knowledge available when throwing last.

Competitive Darts (501 variant).

In this variant both players start with 501 points and alternate 3 throw turns. The player that reduces their score to *exactly zero* first wins the game, but there is a catch - the last throw of the game must hit a double (the double out rule).

If a player hits a score that reduces his total *below zero, exactly 1, or zero but the last throw was not a double*, the result of the turn is a **bust**. The turn ends, and the players score is returned to the value it was at the *beginning of their turn.* Thus, if it is the 2nd or 3rd throw of the turn, the previous throws do not count. An alternate version of the bust rule is to count all the scores up to the throw that went bust. I will start with the latter rule as it is easier to solve (primarily because a player's score never goes up) and the results can be used to help solve the former.

The rules of 501 make the win optimization problem considerably more difficult than the Hi-Score variant because the values of each score can never be simplified down to the 0/0.5/1 split as the number of turns in the game is not defined. There are obviously still 1s (the winning doubles), but never 0s (unless playing against a "perfect" dart player).

To solve this problem, I will first start with a simpler problem:

Minimizing Expected Turns

This can be solved using a similar approach to the Hi-Score variant, except that we start with the lowest possible score (2) and calculate the optimal target for 1, 2, and 3 throws remaining in a turn and then work our way backwards by solving for 3, and then 4, and so on.

Solving For A Score of 2

Solving for 2 is relatively easy, as the optimal strategy will always coincide with the strategy that maximizes the chances to go out on that turn. With the alternate bust rule, we do not have to consider any other possible scores as a bust will return the score to 2.

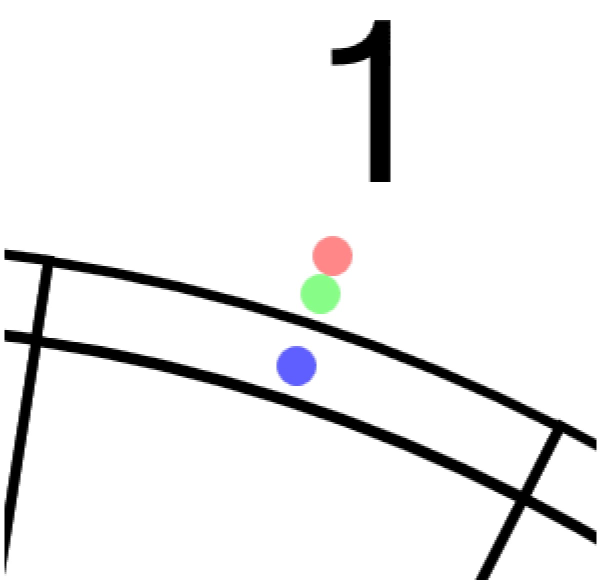
There are three possible results to a single throw:
1. A double 1, winning the game.
2. Any other score on the dartboard, resulting in a bust and ending the turn.
3. A miss (no score), the throw is lost but the turn is not forfeited.

For the last throw of a turn, the latter two produce the same result. Thus, the optimal target is the one that maximizes the chances of #1 - this is, not surprisingly, very close to (but not exactly) the center of the D1.   Pigfooote will hit this about 16.3% of the time and win the game if he does (this is called a “checkout”).

On the 2nd throw of the turn, the equation is not as simple as a bust forfeits the last throw and is less desirable than a miss - which we now know will give the pig a 16.3% of going out on the last throw. Pigfoote should aim slightly outside of the D1 as a result (yes, the pig should be *trying* to miss the board entirely). By doing so, he reduces the odds of hitting the D1 to 15%, but a miss is now over twice as likely as a bust.

The target moves even further out for the 1st throw of the turn, as a bust forfeits two throws instead of one. The foote ends up having a 29.6% chance of ending the game starting from the beginning of the turn, compared to 41.5% if he had 3 throws at the D1 without the possibility of a bust. With the bust rule, Pigfoote would have only a 26.1% of going out on the turn if he aims at the center of the D1 for all throws. The small difference in targets (about 1 cm) results in a relative gain of 13.4% in checkout percentage for the pig.

This shows the optimal target for his 1st, 2nd and 3rd throw.



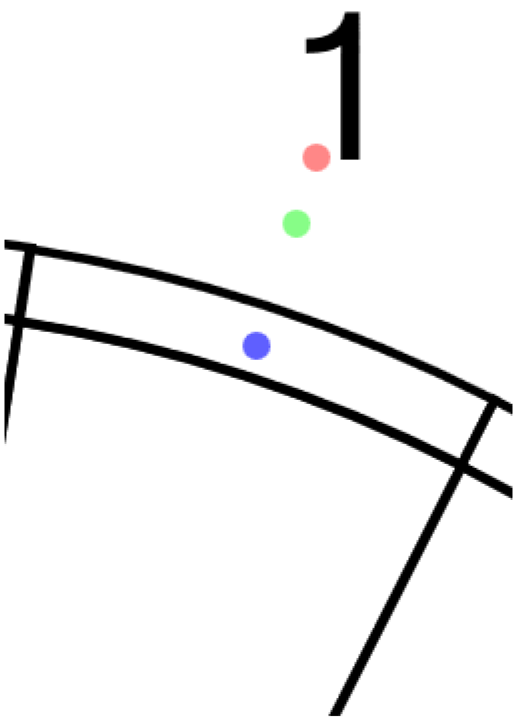
Note that the target for the last throw is a bit closer to the single than a miss due to the curve of the 1, which drops off more vertically on the right side compared to the left.

Throw	Target	Out %	Turns
1	M - 1: H: 0.5 V: 0.14	29.6%	3.38
2	M - 1: H: 0.5 V: 0.06	24.5%	3.55
3	D - 1: H: 0.5 V: 0.44	16.3%	3.83

Throw	D1	Miss	Bust
1	13.3%	66.7%	20.0%
2	15.0%	57.8%	27.2%
3	16.3%	42.1%	41.6%

Note that the average turns to finish before the 1st throw is equal to 1 divided by the odds to checkout on that turn.

This strategy of aiming outside of the center of the double (often off the edge of the board) is frequently recommended - the lower the skill level, the further away the player should aim. The player with double the σ as Pigfoote's should aim far off the board in some cases:



Pigfoote's Turn Distribution Table At the Start of the Turn with a Score of 2

Based on Pigfoote's 29.6% chance of finishing the game on his turn, we can also build a table that shows the odds of Pigfoote finishing in X turns:

Turns	Exact	Cumulative
1	29.63%	29.63%
2	20.85%	50.48%
3	14.67%	65.16%
4	10.32%	75.48%
5	7.27%	82.75%
6	5.11%	87.86%
7	3.60%	91.46%
8	2.53%	93.99%
9	1.78%	95.77%
10	1.25%	97.02%
11	0.88%	97.91%
12	0.62%	98.53%
13	0.44%	98.96%
14	0.31%	99.27%
15	0.22%	99.49%
16	0.15%	99.64%
17	0.11%	99.75%
18	0.08%	99.82%
19	0.05%	99.87%
20	0.04%	99.91%

While this table is *not necessary* in order to solve the current problem of minimizing expected turns, it *will be necessary* when I try to solve the parent problem of maximizing win expectancy.

Solving for a Score of 3+

This is a bit trickier than solving for a score of 2, because the player could start the following turn with either a 2 or a 3 if they do not checkout on this turn. Thus, we can't just assume the strategy that wins on that turn most often is the best strategy. The challenge is that though we know the value of a score of 2, we don't know (yet) the value for a score of 3, as we are trying to solve it now.

To solve this, I took a decidedly programmer's approach (that I will take a couple more times in this article) as opposed to trying to find an elegant mathematical solution.

To begin, I take a guess at the value of a score of 3 at the start of the turn, and use this to find the optimal targets and corresponding values for the 3rd, 2nd and then 1st throws of the turn. This is used to update the value for the 1st throw, which is then used to seed a 2nd pass, which will again update the value. The process continues until both the values and the targets are unchanged after a pass.

The same process is then used to solve for a score of 4, 5, 6, and so on all the way up to 501.

Pigfoote's End of Game Strategy (Score of 2 to 40)

The following table shows Pigfoote's full turn minimization strategy for a score of 2 through 40:

Sc	Th	Target	Out	Turns
2	3	D - 1: H: 0.50 V: 0.44	16.3%	3.83
2	2	M - 1: H: 0.50 V: 0.06	24.4%	3.55
2	1	M - 1: H: 0.50 V: 0.14	29.6%	3.38
3	3	OS - 1: H: 0.50 V: 0.50	0.0%	4.54
3	2	OS - 1: H: 0.50 V: 0.53	11.6%	4.14
3	1	OS - 1: H: 0.50 V: 0.55	17.7%	3.93
4	3	D - 2: H: 0.50 V: 0.86	16.0%	3.66
4	2	D - 2: H: 0.50 V: 0.83	28.5%	3.31
4	1	M - 2: H: 0.50 V: 0.02	36.9%	3.05
5	3	OS - 1: H: 0.50 V: 0.43	0.0%	4.22
5	2	OS - 1: H: 0.50 V: 0.47	11.8%	3.91
5	1	OS - 1: H: 0.50 V: 0.55	21.4%	3.63
6	3	D - 3: H: 0.50 V: 1.00	15.7%	4.02
6	2	M - 3: H: 0.50 V: 0.04	24.1%	3.70
6	1	M - 3: H: 0.50 V: 0.05	31.4%	3.43
7	3	OS - 3: H: 0.50 V: 0.50	0.0%	4.23
7	2	OS - 3: H: 0.50 V: 0.54	11.4%	3.94
7	1	OS - 3: H: 0.50 V: 0.55	20.5%	3.68
8	3	D - 4: H: 0.49 V: 0.64	16.2%	3.48
8	2	D - 4: H: 0.50 V: 0.86	28.5%	3.16
8	1	M - 4: H: 0.50 V: 0.00	37.9%	2.90
9	3	OS - 1: H: 0.50 V: 0.46	0.0%	4.08
9	2	OS - 1: H: 0.50 V: 0.53	11.8%	3.74
9	1	OS - 1: H: 0.50 V: 0.58	21.4%	3.49

10	3	D - 5: H: 0.51 V: 0.90	15.9%	3.84
10	2	M - 5: H: 0.51 V: 0.02	24.1%	3.52
10	1	M - 5: H: 0.51 V: 0.05	31.5%	3.27
11	3	OS - 3: H: 0.50 V: 0.46	0.0%	4.07
11	2	OS - 3: H: 0.50 V: 0.52	11.9%	3.74
11	1	OS - 3: H: 0.50 V: 0.56	21.4%	3.48
12	3	D - 6: H: 0.48 V: 0.88	15.9%	3.72
12	2	M - 6: H: 0.52 V: 0.00	28.4%	3.39
12	1	M - 6: H: 0.53 V: 0.02	37.3%	3.12
13	3	OS - 5: H: 0.51 V: 0.45	0.0%	4.09
13	2	OS - 5: H: 0.50 V: 0.53	11.6%	3.77
13	1	OS - 5: H: 0.50 V: 0.59	21.1%	3.51
14	3	D - 7: H: 0.50 V: 0.91	15.9%	3.87
14	2	M - 7: H: 0.50 V: 0.01	24.1%	3.54
14	1	M - 7: H: 0.50 V: 0.05	31.4%	3.28
15	3	OS - 7: H: 0.50 V: 0.49	0.0%	4.08
15	2	OS - 7: H: 0.50 V: 0.53	11.6%	3.77
15	1	OS - 7: H: 0.50 V: 0.55	20.6%	3.53
16	3	D - 8: H: 0.44 V: 0.64	16.1%	3.41
16	2	D - 8: H: 0.48 V: 0.76	28.5%	3.08
16	1	D - 8: H: 0.50 V: 0.92	38.2%	2.82
17	3	OS - 1: H: 0.50 V: 0.46	0.0%	3.99
17	2	OS - 1: H: 0.50 V: 0.53	11.8%	3.66
17	1	OS - 1: H: 0.50 V: 0.58	21.4%	3.40
18	3	D - 9: H: 0.47 V: 0.91	15.8%	3.74
18	2	D - 9: H: 0.47 V: 0.99	24.6%	3.41

18	1	M - 9: H: 0.47 V: 0.01	32.7%	3.13
19	3	OS - 3: H: 0.54 V: 0.46	0.0%	3.97
19	2	OS - 3: H: 0.38 V: 0.47	14.2%	3.58
19	1	OS - 3: H: 0.36 V: 0.50	24.9%	3.28
20	3	D - 10: H: 0.47 V: 0.92	15.9%	3.62
20	2	D - 10: H: 0.46 V: 0.90	28.8%	3.28
20	1	D - 10: H: 0.44 V: 0.98	38.2%	3.01
21	3	OS - 5: H: 0.55 V: 0.50	0.0%	3.99
21	2	OS - 5: H: 0.45 V: 0.52	11.7%	3.62
21	1	OS - 5: H: 0.41 V: 0.54	22.8%	3.32
22	3	D - 11: H: 0.54 V: 0.88	15.9%	3.72
22	2	D - 11: H: 0.53 V: 0.94	24.8%	3.39
22	1	M - 11: H: 0.51 V: 0.00	33.1%	3.11
23	3	OS - 7: H: 0.28 V: 0.48	0.0%	3.95
23	2	OS - 7: H: 0.30 V: 0.46	14.5%	3.54
23	1	OS - 7: H: 0.30 V: 0.48	26.5%	3.22
24	3	D - 12: H: 0.52 V: 0.84	16.0%	3.55
24	2	D - 12: H: 0.51 V: 0.88	28.7%	3.22
24	1	D - 12: H: 0.51 V: 0.93	39.1%	2.94
25	3	OS - 17: H: 0.59 V: 0.44	0.0%	3.99
25	2	OS - 17: H: 0.64 V: 0.45	14.1%	3.60
25	1	OS - 17: H: 0.63 V: 0.45	25.2%	3.29
26	3	D - 13: H: 0.51 V: 0.96	15.8%	3.72
26	2	D - 13: H: 0.50 V: 0.98	25.0%	3.39
26	1	M - 13: H: 0.50 V: 0.01	33.4%	3.11
27	3	OS - 19: H: 0.29 V: 0.43	0.0%	3.94

27	2	OS - 19: H: 0.32 V: 0.43	15.1%	3.55
27	1	OS - 19: H: 0.16 V: 0.44	26.9%	3.24
28	3	D - 14: H: 0.52 V: 0.88	16.0%	3.63
28	2	D - 14: H: 0.50 V: 0.92	28.6%	3.30
28	1	D - 14: H: 0.51 V: 1.00	37.9%	3.02
29	3	OS - 13: H: 0.53 V: 0.42	0.0%	3.96
29	2	OS - 13: H: 0.51 V: 0.46	11.5%	3.59
29	1	OS - 13: H: 0.52 V: 0.49	23.9%	3.25
30	3	D - 15: H: 0.51 V: 0.95	15.8%	3.73
30	2	D - 15: H: 0.51 V: 0.95	24.9%	3.39
30	1	M - 15: H: 0.51 V: 0.01	33.4%	3.11
31	3	OS - 15: H: 0.52 V: 0.51	0.0%	3.95
31	2	OS - 15: H: 0.50 V: 0.49	11.5%	3.59
31	1	OS - 15: H: 0.52 V: 0.49	23.0%	3.27
32	3	D - 16: H: 0.53 V: 0.68	16.1%	3.34
32	2	D - 16: H: 0.55 V: 0.70	29.3%	3.01
32	1	D - 16: H: 0.54 V: 0.73	40.0%	2.73
33	3	OS - 17: H: 0.58 V: 0.47	0.0%	3.93
33	2	OS - 1: H: 0.50 V: 0.50	11.8%	3.54
33	1	OS - 1: H: 0.50 V: 0.52	24.3%	3.21
34	3	D - 17: H: 0.45 V: 0.88	15.9%	3.65
34	2	D - 17: H: 0.45 V: 0.96	24.7%	3.32
34	1	D - 17: H: 0.44 V: 0.98	33.1%	3.04
35	3	OS - 3: H: 0.76 V: 0.43	0.0%	3.80
35	2	OS - 3: H: 0.77 V: 0.43	15.3%	3.42
35	1	OS - 3: H: 0.73 V: 0.45	27.6%	3.09

36	3	D - 18: H: 0.52 V: 0.80	16.0%	3.50
36	2	D - 18: H: 0.54 V: 0.83	29.0%	3.17
36	1	D - 18: H: 0.54 V: 0.88	38.8%	2.89
37	3	OS - 5: H: 0.48 V: 0.50	0.0%	3.89
37	2	OS - 5: H: 0.50 V: 0.49	11.9%	3.52
37	1	OS - 5: H: 0.48 V: 0.51	24.7%	3.18
38	3	D - 19: H: 0.48 V: 0.86	16.0%	3.61
38	2	M - 19: H: 0.48 V: 0.01	24.5%	3.30
38	1	D - 19: H: 0.47 V: 0.98	33.3%	3.01
39	3	OS - 7: H: 0.40 V: 0.46	0.0%	3.84
39	2	OS - 7: H: 0.38 V: 0.46	14.2%	3.47
39	1	OS - 7: H: 0.40 V: 0.48	27.0%	3.14
40	3	D - 20: H: 0.50 V: 0.75	16.1%	3.45
40	2	D - 20: H: 0.50 V: 0.81	28.7%	3.13
40	1	D - 20: H: 0.50 V: 0.88	39.4%	2.85

Note that the table gets generally redder at the end - in this score range, there is no inherent value in a score closer to the ultimate target score of 0.

This shows the strategy and values at the beginning of each turn, sorted by the expected turns:

Sc	Target	Out	Turns
32	D - 16: H: 0.54 V: 0.73	40.0%	2.73
16	D - 8: H: 0.50 V: 0.92	38.2%	2.82
40	D - 20: H: 0.50 V: 0.88	39.4%	2.85
36	D - 18: H: 0.54 V: 0.88	38.8%	2.89
8	M - 4: H: 0.50 V: 0.00	37.9%	2.90

24	D - 12: H: 0.51 V: 0.93	39.1%	2.94
20	D - 10: H: 0.44 V: 0.98	38.2%	3.01
38	D - 19: H: 0.47 V: 0.98	33.3%	3.01
28	D - 14: H: 0.51 V: 1.00	37.9%	3.02
34	D - 17: H: 0.44 V: 0.98	33.1%	3.04
4	M - 2: H: 0.50 V: 0.02	36.9%	3.05
35	OS - 3: H: 0.73 V: 0.45	27.6%	3.09
30	M - 15: H: 0.51 V: 0.01	33.4%	3.11
26	M - 13: H: 0.50 V: 0.01	33.4%	3.11
22	M - 11: H: 0.51 V: 0.00	33.1%	3.11
12	M - 6: H: 0.53 V: 0.02	37.3%	3.12
18	M - 9: H: 0.47 V: 0.01	32.7%	3.13
39	OS - 7: H: 0.40 V: 0.48	27.0%	3.14
37	OS - 5: H: 0.48 V: 0.51	24.7%	3.18
33	OS - 1: H: 0.50 V: 0.52	24.3%	3.21
23	OS - 7: H: 0.30 V: 0.48	26.5%	3.22
27	OS - 19: H: 0.16 V: 0.44	26.9%	3.24
29	OS - 13: H: 0.52 V: 0.49	23.9%	3.25
10	M - 5: H: 0.51 V: 0.05	31.5%	3.27
31	OS - 15: H: 0.52 V: 0.49	23.0%	3.27
19	OS - 3: H: 0.36 V: 0.50	24.9%	3.28
14	M - 7: H: 0.50 V: 0.05	31.4%	3.28
25	OS - 17: H: 0.63 V: 0.45	25.2%	3.29
21	OS - 5: H: 0.41 V: 0.54	22.8%	3.32
2	M - 1: H: 0.50 V: 0.14	29.6%	3.38

17	OS - 1: H: 0.50 V: 0.58	21.4%	3.40
6	M - 3: H: 0.50 V: 0.05	31.4%	3.43
11	OS - 3: H: 0.50 V: 0.56	21.4%	3.48
9	OS - 1: H: 0.50 V: 0.58	21.4%	3.49
13	OS - 5: H: 0.50 V: 0.59	21.1%	3.51
15	OS - 7: H: 0.50 V: 0.55	20.6%	3.53
5	OS - 1: H: 0.50 V: 0.55	21.4%	3.63
7	OS - 3: H: 0.50 V: 0.55	20.5%	3.68
3	OS - 1: H: 0.50 V: 0.55	17.7%	3.93

The most desirable score for Pigfoote is 32 - assuming Pigfoote (who is fairly skilled, remember) stays within the pie slice, he will get at least 5 consecutive throws at a double without going bust.

As you would expect, the odd scores suffer because at least 2 throws are needed to end the game. The **35** is an exception as the 12th best score (ranked above 9 of the 20 even numbers):

- The optimal target of a 3 would reduce the score to the most desired score (32).
- A miss to the left is a 19, which would reduce the score to the 2nd most desired score (16)
- A miss to the right is a 17, which leaves 18 - not the best even number, but still better than any of the other 19 odd numbers.

Because of the 2nd and 3rd points, Pigfoote should aim on the left side of the 3, about ¼ of the way from the 19 to the 17.

The **6** is Pigfoote's worst even score. It beats only 7 odd scores by average turns, all of which Pigfoote would have at least a 10% lower chance to go out in one turn compared to the 6.

The following summarizes Pigfoote's end of game strategy:

- On an even score, he should aim somewhere between the center of the winning double and just off the edge of the board depending on the following:
  - How bad a single of the target is relative to a miss. Generally, he should aim closer to the center for numbers divisible by four compared to those that are not.

    - The throw number - Pigfoote should be slightly more aggressive on the last throw of his turn, though the average distance between the 1st and 3rd throws is only about 1 mm.

  Note that both extremes occur at a score of 2 - for all other even scores, he should never aim further outside his score 2/throw 1 target, and he should always aim further outside than his score 2/throw 3 target.
- For odd scores, Pigfoote should go for the single that leaves the highest power of 2 remaining (32, 16, 8, 4, 2), with the two exceptions taking advantage of the consecutive odd numbers 19, 3 and 17:
    - A score of 27, where the strategy of aiming at the 19 close to the 3 and the near-certainty of reaching the 6th or 7th (8 and 24) best scores beats the high probability of reaching the 2nd best score (16) by aiming at the 11.
    - A score of 33 on the last throw of the turn only, where a target of 17 (close to the center, but on the 3 side) edges the default target of 1.

Pigfoote's Setup Throw Strategy (Score: 41 to 60)

Pigfoote's strategy to minimize turns in this score range contains no surprises and can be almost entirely explained by two rules:

1. With a score below 52, aim for the single that leaves a score of 32
2. Otherwise, aim for the single that leaves a score of 40

For almost all scores, he should aim at roughly the middle of the outer single sector. The two exceptions are a score of 42 and 48 - where he should still aim at the 10 and 16, but at the ¼ mark closer to the 6 and 8 as both of those scores also leave a "good" double. For this reason, 42 and 48 are also the two best scores in this range for the pig, both in terms of average turns and the chances to go out on that turn.

My program also creates an alternate strategy (which I call the "win now" strategy) for Pigfoote that maximizes the chance to go out on that turn without caring about average turns.

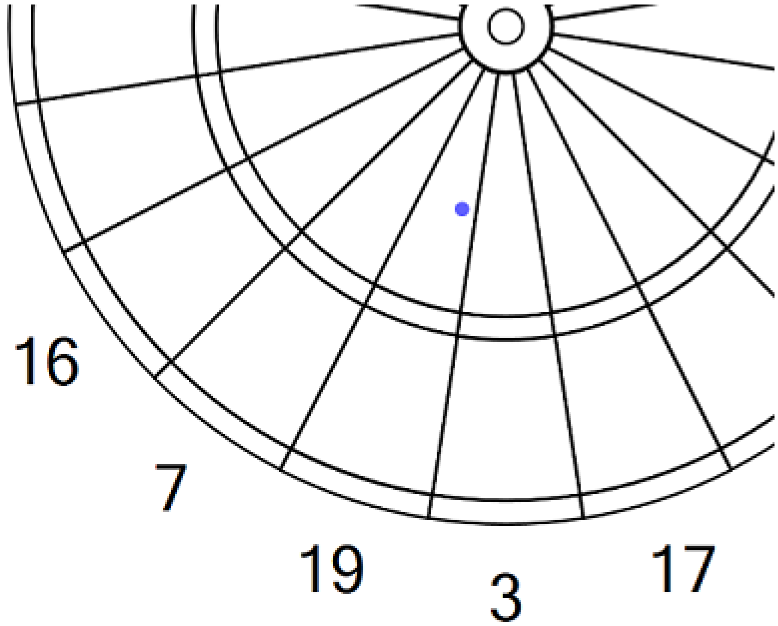
The following chart shows both strategies for Pigfoote for the 1st throw of the turn (for the turn strategy, there is rarely any significant difference for the 2nd and 3rd throws).

Sc	Target	Out	Turns	WN Target	Out	Turns
41	OS 9 (0.49,0.52)	24.8%	3.17	OS 17 (0.73,0.48)	26.1%	3.34
42	**OS 10 (0.29,0.46)**	**28.2%**	**3.10**	OS 6 (1.00,0.34)	29.0%	3.15
43	OS 11 (0.43,0.49)	24.8%	3.16	**IS 19 (0.26,0.58)**	**28.1%**	**3.25**

44	OS 12 (0.48,0.49)	25.9%	3.12	OS 8 (0.12,0.52)	28.4%	3.30
45	OS 13 (0.52,0.51)	24.9%	3.16	OS 13 (0.50,0.44)	25.7%	3.22
46	OS 14 (0.46,0.50)	25.5%	3.13	OS 10 (0.02,0.34)	28.8%	3.21
47	OS 15 (0.51,0.49)	24.5%	3.16	OS 7 (0.11,0.44)	27.6%	3.29
48	**OS 16 (0.74,0.61)**	**27.7%**	**3.08**	OS 8 (0.03,0.53)	28.3%	3.13
49	OS 17 (0.51,0.51)	24.1%	3.17	OS 13 (0.51,0.43)	25.6%	3.34
50	OS 18 (0.51,0.55)	24.6%	3.16	OS 14 (0.50,0.50)	26.1%	3.32
51	OS 19 (0.48,0.51)	24.4%	3.16	OS 11 (0.50,0.44)	25.4%	3.31
52	OS 20 (0.50,0.57)	24.8%	3.16	OS 16 (0.52,0.52)	26.2%	3.33
53	OS 13 (0.51,0.48)	24.3%	3.24	OS 17 (0.51,0.45)	25.0%	3.35
54	OS 14 (0.49,0.51)	24.8%	3.23	OS 14 (0.50,0.50)	25.8%	3.29
55	OS 15 (0.47,0.49)	24.2%	3.25	OS 15 (0.47,0.45)	25.0%	3.32
56	OS 16 (0.56,0.45)	25.2%	3.21	OS 16 (0.55,0.52)	25.6%	3.25
57	OS 17 (0.51,0.51)	24.3%	3.26	OS 17 (0.51,0.46)	24.7%	3.31
58	OS 18 (0.51,0.50)	24.7%	3.25	OS 18 (0.50,0.49)	25.1%	3.30
59	OS 19 (0.53,0.46)	24.3%	3.24	OS 19 (0.51,0.45)	24.8%	3.30
60	OS 20 (0.50,0.61)	24.5%	3.26	OS 20 (0.50,0.63)	24.8%	3.30
Avg		**25.0%**	**3.18**		**26.3%**	**3.28**

For the most part, the differences between the two targets can be summed up by this: the "win now" strategy wants to end up at *any double* while the turn strategy wants to end up on the *best double*. Thus the 2nd target will seek out consecutive odd or even numbers that both give a chance to win on the next throw (e.g., 17/3 at 41, 6/10 at 42, and so on).

The one target that really surprised me for the win now strategy is the Inner Single 19 with a score of 43, both because of the target and because it has the highest gain in checkout percentage of any win now target (28.13% vs. 24.8%):



Although it takes advantage of 4 consecutive odd numbers, it wasn't clear to me why this target is superior to a similar target in the outer sector - so I looked into it.

First, there are categories of results that matter for the win now strategy starting from a score of 43, listed in order of preference:

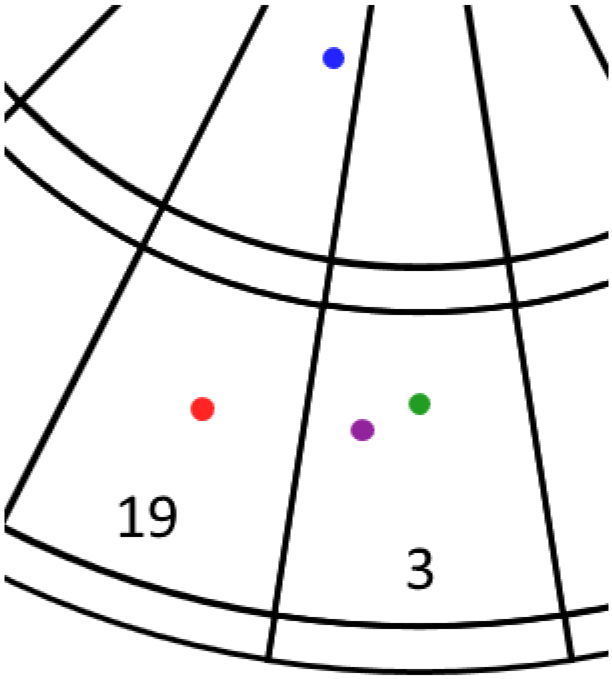
1. Score is reduced to a multiple of 4. (7, 19, and 3 accomplish this)
2. Score is reduced to a multiple of 2, but not 4. (17, T7, T3)
3. Score is reduced (or remains on) an odd number. (16, D7/19/3)
4. A Bust (T19, T17, T16)

The following table shows Pigfoote's average odds of going out for each category (there is a slight variance depending on the actual score):

Category	Out %
1	29.3%
2	25.2%
3	15.7%
4	0.0%

I looked at all of the possible targets in the 7 to 17 zone but in the outer single sector to compare it to the above target. While the preferred target doesn't fare the best for any of the four categories, it does well in all of them and comes up on top as a result.

Here it is again compared to three selected targets in the outer zone:



The **red** dot maximizes Pigfoote's odds of landing on a multiple of 4, the **green** dot maximizes Pigfoote's odds of landing on a multiple of 2, and the **purple** dot scores the best of the outer single targets for the win now strategy.

Here's a chart comparing the four targets:

	Blue	**Red**	**Green**	**Purple**
Mult of 4	88.32%	91.71%	78.57%	87.97%
Mult of 2	95.77%	93.14%	96.04%	95.31%
Odd	3.29%	3.34%	1.39%	2.65%
Bust	0.94%	3.52%	2.57%	2.04%
Out %	**28.13%**	**27.41%**	**27.34%**	**27.67%**

As shown, the red and **green** targets are giving up too much in the other categories to maximize their own. The **purple** target doesn't have any real advantage over the original, blue dot (trading odd scores for busts does not count as an advantage).

The inner single sector in fact never shows up in Pigfoote's turn optimization tables, which cover all 1500 score/throw combinations, though it makes 17 other appearances in his win now strategy table.

Pigfoote's Pre-Setup Throw Strategy (Score: 61 to 80)

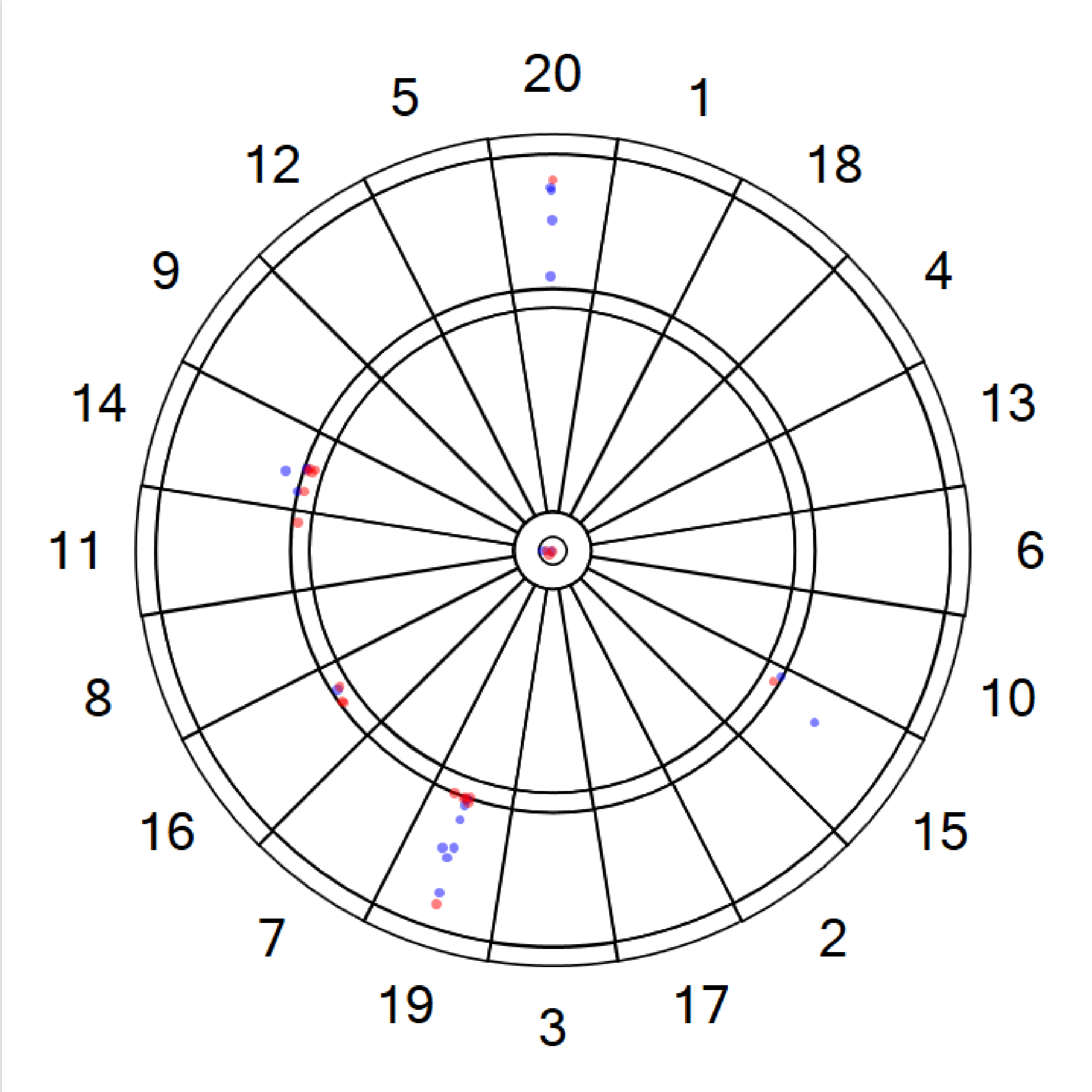
Now that it is no longer possible to hit a single and have a chance to checkout the next throw (unless you go for a double bull), Pigfoote's target strategy is much less obvious. Here is the table:

Sc	Target	Out	Turns	WN Target	Out	Turns
61	DB 7 (1.00,0.11)	17.1%	3.43	DB 3 (0.50,0.08)	18.2%	3.55
62	OS 14 (0.42,0.12)	15.4%	3.43	T 14 (0.54,0.69)	16.4%	3.52
63	OS 15 (0.35,0.38)	13.9%	3.47	DB 7 (0.55,0.39)	16.2%	3.61
64	**T 16 (0.67,0.73)**	**16.3%**	**3.40**	T 16 (0.69,0.52)	16.8%	3.46
65	DB 11 (0.50,0.63)	15.9%	3.46	DB 11 (0.50,0.39)	16.8%	3.58
66	T 14 (0.51,0.85)	14.1%	3.48	T 14 (0.54,0.43)	14.8%	3.67
67	OS 19 (0.63,0.41)	13.8%	3.48	OS 19 (0.72,0.00)	15.1%	3.55
68	OS 20 (0.49,0.51)	12.9%	3.50	T 14 (0.49,0.49)	14.5%	3.65
69	T 15 (0.11,0.91)	13.4%	3.54	T 15 (0.21,0.71)	14.1%	3.63
70	OS 14 (0.22,0.00)	13.5%	3.54	T 14 (0.24,0.66)	14.1%	3.60
71	OS 19 (0.55,0.47)	10.1%	3.57	**T 11 (0.85,0.70)**	**13.6%**	**3.67**
72	OS 20 (0.48,0.75)	10.9%	3.55	T 16 (0.52,0.84)	12.8%	3.65
73	OS 19 (0.56,0.01)	11.3%	3.59	T 19 (0.53,0.95)	12.8%	3.71
74	T 14 (0.53,0.81)	12.1%	3.58	T 14 (0.50,0.77)	12.8%	3.65

75	OS 19 (0.52,0.38)	10.3%	3.62	OS 19 (0.60,0.01)	11.8%	3.70
76	OS 20 (0.47,0.09)	10.8%	3.61	T 16 (0.50,0.83)	12.6%	3.70
77	OS 19 (0.56,0.06)	11.2%	3.64	T 19 (0.59,0.97)	12.2%	3.72
78	OS 19 (0.52,0.73)	10.7%	3.63	OS 19 (0.51,0.82)	11.4%	3.70
79	OS 19 (0.56,0.17)	10.5%	3.66	OS 19 (0.53,0.03)	11.1%	3.74
80	OS 20 (0.49,0.73)	11.0%	3.65	OS 20 (0.50,0.81)	11.4%	3.71
Avg		**12.8%**	**3.54**		**14.0%**	**3.64**

64 is the 48 of this score range (the most desirable score), while 71 is is the 43 (the score where the win now strategy makes the biggest difference). A full 12 out of the 20 win now targets are in the triple zones compared to just 4 for the turn targets. The difference isn't as much as it sounds though, as several of the turn targets are just outside the triple.

The following shows the target points on the board (blue = turn strategy, red = win now strategy)



The differences between the two strategies fall into two categories:

1. A slightly more aggressive approach for the win now strategy (e.g., aiming closer to the center of a triple) - 13 out of 20
2. A totally different target - 7 out of 20
    a. Scores of 67 and 75 fall somewhere in between, but I included them in this category as while the targets are in the same sector, the win now target is very close to the triple while the turn strategy is not.

The Peak of the Win Now Strategy’s Checkout Advantage

The following table shows the targets for a score of 81-90 at the start of the turn, where the difference in checkout percentage is greatest between the two:

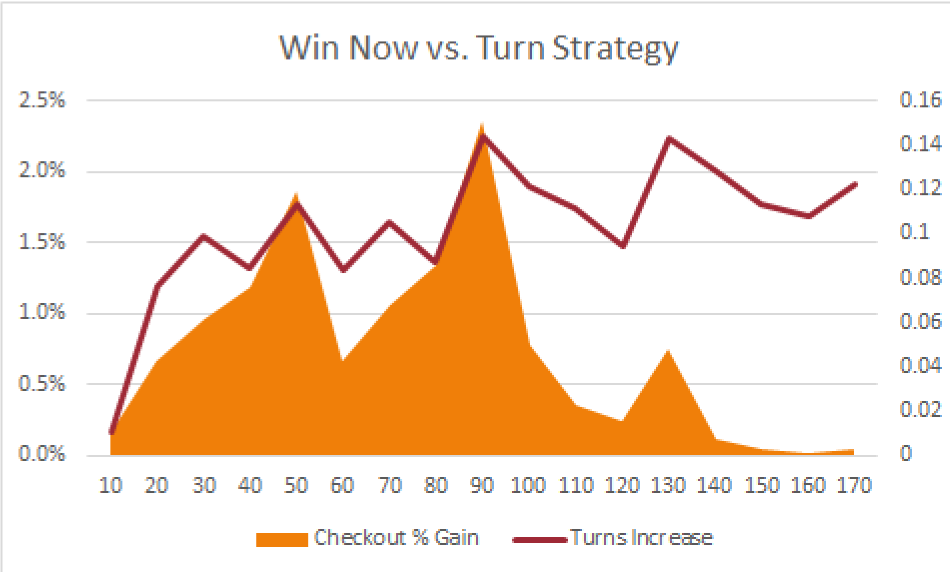
Sc	Target	Out	Turns	WN Target	Out	Turns
81	T 19 (0.64,0.64)	5.5%	3.77	T 19 (0.58,0.55)	8.0%	3.92
82	DB 7 (1.00,0.11)	6.7%	3.76	DB 6 (0.50,0.00)	8.1%	3.90
83	OS 19 (0.56,0.01)	5.0%	3.79	T 17 (0.52,0.67)	7.6%	3.93
84	OS 20 (0.43,0.01)	5.4%	3.79	T 16 (0.57,0.56)	8.0%	3.91
85	DB 14 (0.52,0.25)	5.7%	3.81	T 15 (0.44,0.65)	7.7%	3.96
86	T 18 (0.55,0.84)	5.0%	3.83	T 18 (0.49,0.65)	7.5%	3.95
87	T 19 (0.62,0.87)	4.4%	3.85	T 17 (0.51,0.69)	7.5%	4.00
88	T 16 (0.60,0.63)	5.0%	3.83	T 20 (0.50,0.69)	7.4%	3.99
89	T 19 (0.57,0.73)	5.1%	3.84	T 19 (0.52,0.67)	7.6%	3.98
90	T 18 (0.53,0.85)	5.1%	3.89	T 20 (0.50,0.69)	7.0%	4.05
Avg		5.3%	3.82		7.6%	3.96

The checkout percentage gain of 2.3% is almost twice that as the range of 41-60 (1.3%) and 61-80 (1.2%). Relatively, the gain is far greater, going from about about a 5 to 10 to **43** percent increase over the checkout odds of the turn strategy.

Note that the target is roughly the same for 4 out of the 10 scores yet the numbers are still quite different - the follow up targets will presumably be quite different for the two strategies.

The drop off in checkout percentage gain is precipitous after a score of 90 - with the highest being a 1.18% gain at 91, and 94 being the last score above 1%.

The advantage of the turn based strategy - while also peaking in this range, is relatively much more stable as shown by the following chart:



This is all based on the strategy for the 1st throw of a turn. I would be remiss if I did not mention the single score/throw combination where the difference is by far the greatest. The situation should come as no surprise to anyone who has played or followed competitive darts:

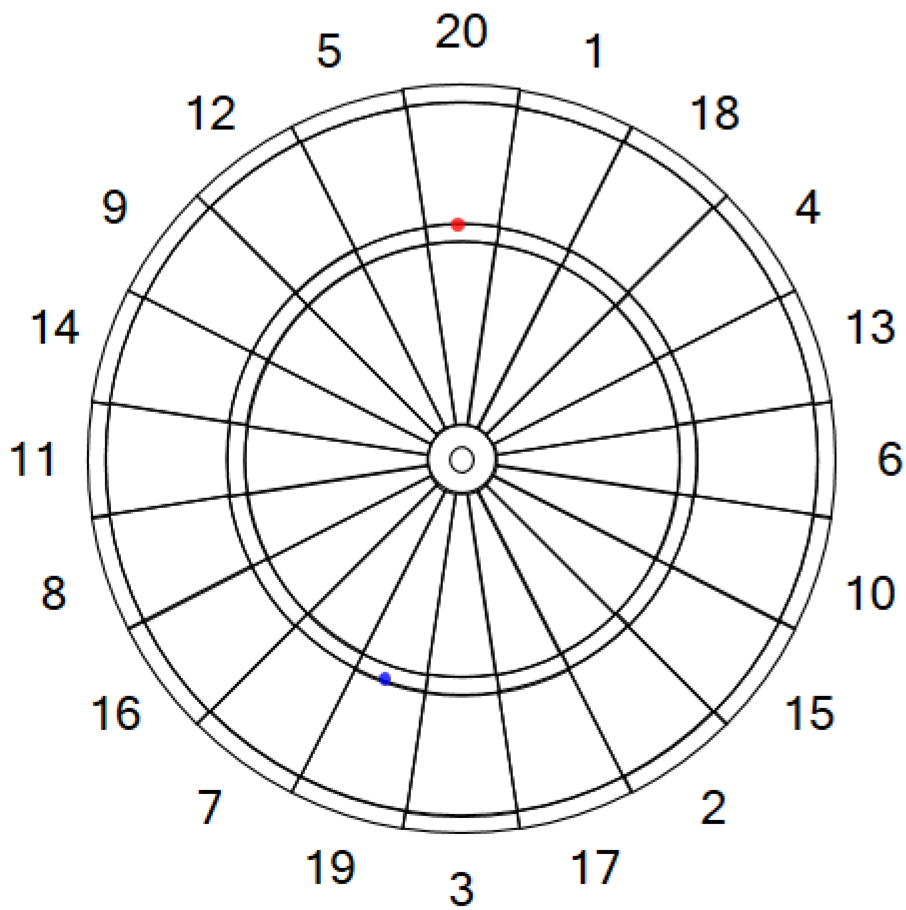
	Turn	Win Now
Target	OS 18 (0.51,0.47)	DB 6 (0.50,0.00)
Out %	**0.0%**	**6.72%**
Turns	3.85	4.01

Of course, the situation is the last throw of a turn with a score of 50.

The Rest of Pigfoote's Strategy

Above a score of 90, 104 (where the T16 is recommended) is the only time Pigfootes turn strategy recommends a target other than (or at least extremely close to) Pigfoote's best two scoring targets: the Triple 19 and the Triple 20. And though the win now strategy occasionally has Pigfoote chasing low probability 3 throw checkouts, this strategy is gaining very little at these scores.

A score of 160 is the last time that the target is on the border of the triple and outer single sectors, and the minute differences in targets are completely gone by a score of 250. The one big difference in targets still exists, however:



Although Pigfoote's optimal scoring target is the T19, his σ of 17mm is just above the threshold of 16.4 mm where the target shifts from the T20 to the T19. He is only giving up about 0.06 points per turn by aiming at the red dot compared to the blue.

Thus, there are certain scores as high as 460 where the strategy recommends the T20.

- Every multiple of 20 up to 440 (260, 280… 460)
- Every score 1 higher than a multiple of 20 up to 341 (261, 281, 301, 321, 341)
- Every score 5 above a multiple of 20 up to 365 (265, 285, 305, 325, 345, 365)
- A score of 385 on the 2nd throw of a turn only
- A score of 405 on the 1st throw of a turn only
- A score of 460 on the 1st or 2nd throw of a turn (but not the 3rd).

These recommendations show the relative increase in value of a score divisible of 20. The strategy recommends the 20 for some scores 1 and 5 above a multiple of 20 because the 20 is next to the 1 and the 5, and thus there is a good chance of eventually landing on a multiple of 20. Of course, Pigfoote shouldn't *try* to hit a 1 or a 5 as he would give up far too much in terms of scoring average by doing so.

The following table shows the count of each target for the 1233 score/throw combinations above 90, grouped by sector:

Target	Num	Count	High
T19/OS19	1	1	119
T16	2	2	104
OS20 (V=0.01)	2	2	136
T20/OS20	2	20	160
T20	9	175	460
T19	21	1033	501
Total	**37**	**1233**	

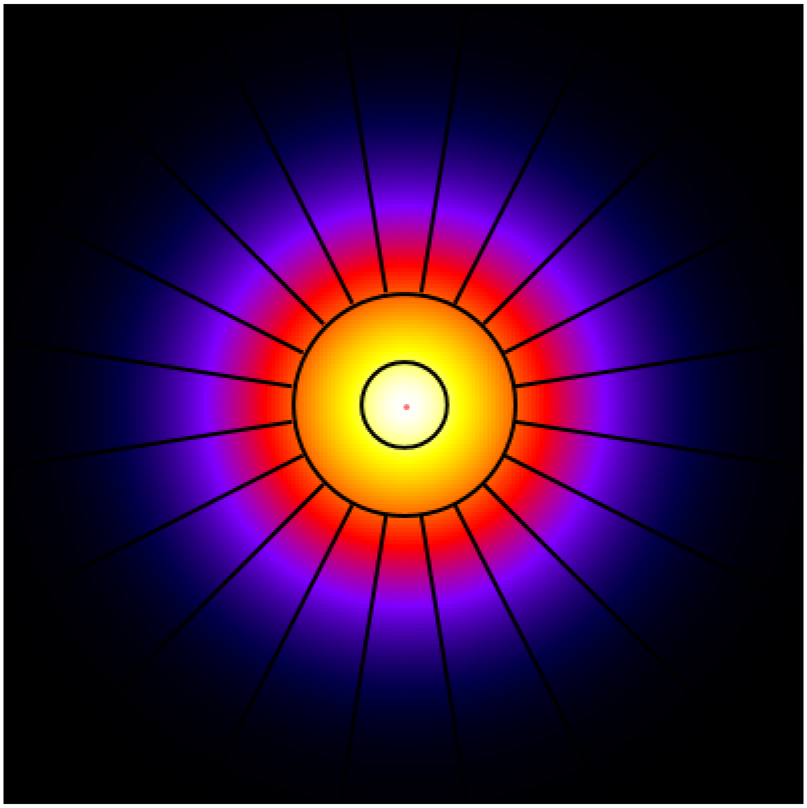
The T/OS targets are on the border between the triple and outer single sectors. The "Num" column shows the number of specific targets for each sector. While technically there are 21 different targets recommended within the T19, the range is quite small as no target is more than 2.5mm from any other target (and all are within 1.5 mm of the blue dot).

Pigfoote should average **10.78** turns to finish from a starting score of 501 if he follows the turn minimization strategy, and **10.89** turns if he follows the win now strategy.

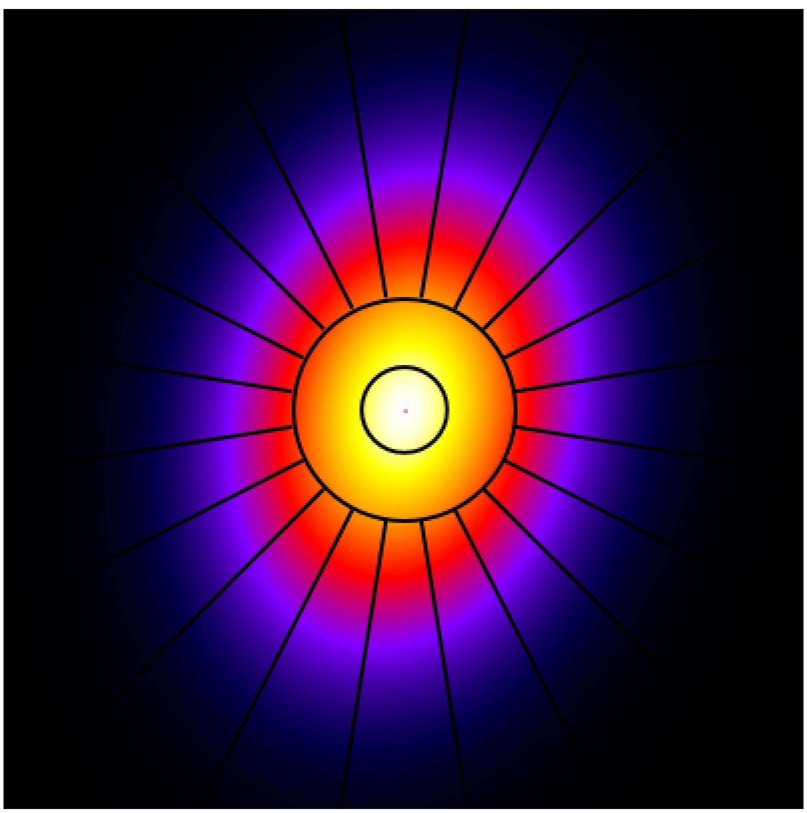
Pigfoote vs. Duck vs. Goose

While I showed the probability density functions for Pigfoote and Duck above, I will repost them here as well as for another player, Goose, who throws the opposite* of Duck:

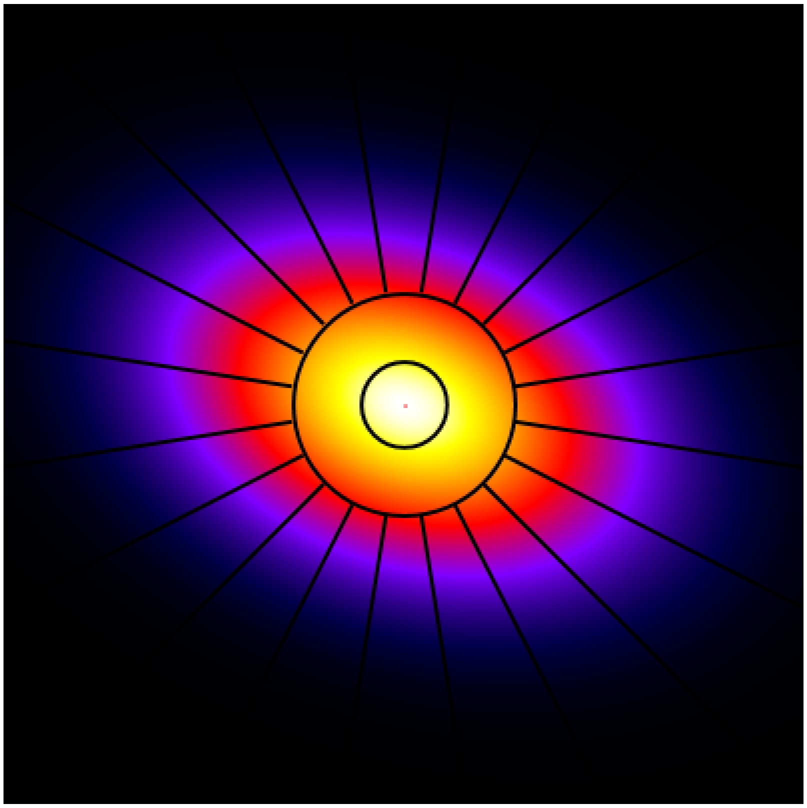
Pigfoote



Duck



Goose



While Pigfoote's radially symmetrical pdf means that he can hit each double at almost exactly the same rate (16.3%), this is not the case for Duck and Goose, whose double percent ranges from about 14.2% to as high as 18.4%.

This shows how the 3 players compare at the end of the game in terms of expected turns to finish:

Score	Pigfoote	Duck	Goose
2	3.38	3.79	3.04
3	3.93	4.30	3.69
4	3.05	3.34	2.86
5	3.63	3.87	3.56
6	3.43	3.82	3.16
7	3.68	3.94	3.58

8	2.90	3.14	2.76
9	3.49	3.68	3.45
10	3.27	3.52	3.19
11	3.48	3.68	3.42
12	3.12	3.32	3.01
13	3.51	3.76	3.38
14	3.28	3.56	3.08
15	3.53	3.76	3.37
16	2.82	3.01	2.73
17	3.40	3.54	3.25
18	3.13	3.25	3.17
19	3.28	3.49	3.16
20	3.01	3.13	3.03
21	3.32	3.51	3.21
22	3.11	3.20	3.10
23	3.22	3.45	3.09
24	2.94	3.12	2.89
25	3.29	3.50	3.16
26	3.11	3.24	3.07
27	3.24	3.45	3.10
28	3.02	3.16	2.96
29	3.25	3.47	3.16
30	3.11	3.20	3.12
31	3.27	3.48	3.18
32	2.73	2.89	2.66

33	3.21	3.37	3.15
34	3.04	3.20	2.97
35	3.09	3.27	3.01
36	2.89	3.05	2.82
37	3.18	3.36	3.11
38	3.01	3.28	2.81
39	3.14	3.30	3.09
40	2.85	3.02	2.78
Avg	**3.21**	**3.42**	**3.11**

Duck's throwing does not suit the finish very well. His worst doubles are the D1/D19, which means he does quite poorly with a score of 2. And since a score of 2 is inherently factored into the expected turns for every other score, his average turns is the highest of the three players. Conversely, the D1 is one of Goose's best doubles, and he does the best of the three here.

Unfortunately for Goose, his throwing does not suit high scoring, where aiming at the 20 and 19 will result in a lot of misses to the low scores on the left and right sides (though, he will hit a few more triples than Pigfoote or Duck). Meanwhile Duck's throwing suits these targets very well.

Pigfoote has no weaknesses, but also no strengths. In this variant of darts where you are allowed to pick your target, this ends up costing him a bit strategically compared to his feathered competition.

Although he can expect to suffer a few embarrassing defeats while stuck on a score of two and unable to finish, **Duck** ends up being the clear winner in this battle of fantasy dart players named after farm animals:

	Player	Turns	Win Pct
	Duck	10.75	52.1%
	Pigfoote	10.78	50.2%

	Goose	10.88	47.7%

Duck is much more likely to take over 20 turns to finish, which adds about 0.15 turns to his average with little effect on winning percentage since the game likely won't last that long.

*- I gave Goose an hσ of 20mm and vσ of 14 mm, compared to Duck's hσ of 15mm and vσ of 20mm to make this match more interesting (otherwise, Goose loses his only advantage at the end of the game).

First Player Advantage

This shows the advantage of the 1st player in this contest:

	1st	2nd
Pigfoote	56.5%	44.0%
Duck	58.3%	45.9%
Goose	53.9%	41.4%
Avg	**56.3%**	**43.7%**

Pigfoote vs. Michael van Gerwen

In this section I will present a few scenarios where Pigfoote must make a decision in a hypothetical match against the current top ranked dart player in the world, Michael van Gerwen (MvG).

Although a legend of my fantasy dart leagues, Pigfoote is at a large disadvantage in this match:

	Pigfoote	**MvG***
σ	17mm	6.6mm
Scoring	59.2	112
Checkout	15%	45%
Avg Turns	10.78	5.35

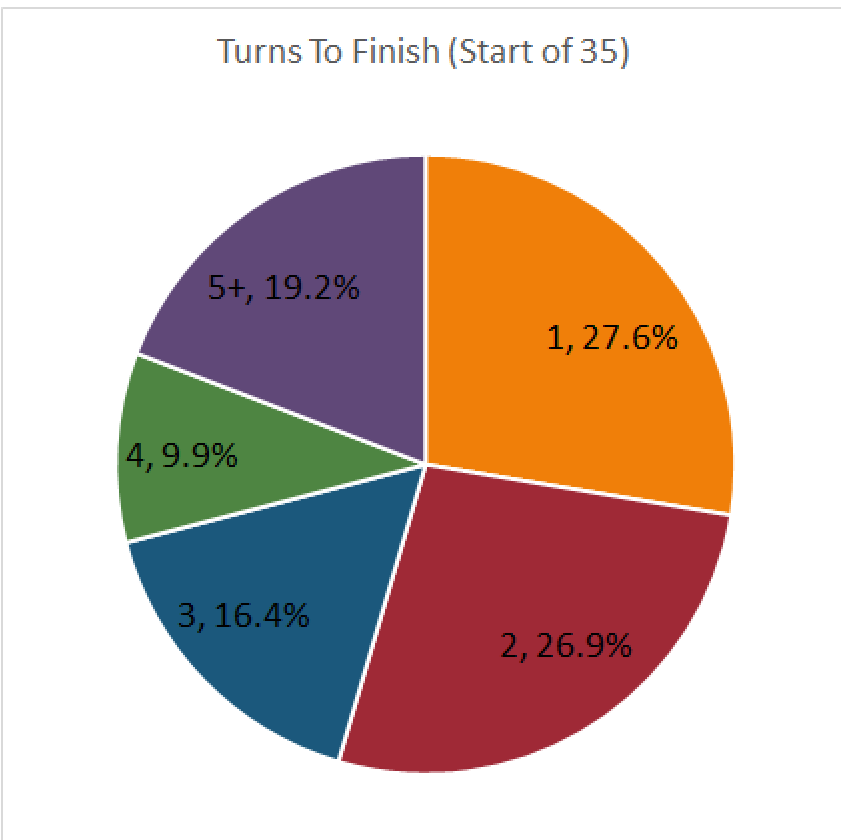
* -Besides checkout percentage, MvG's numbers are estimated as I was unable to find exact statistics online. While MvG's career 3 dart average is about 102, this includes throws at the end of the game and he will score higher than that when aiming at his best scoring target.

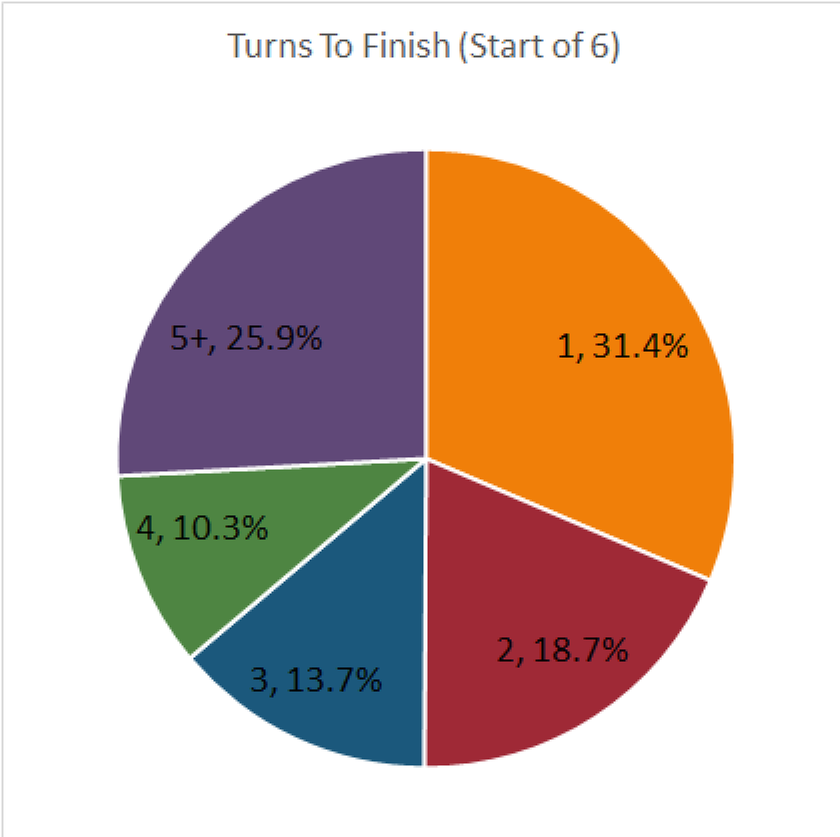
Scenario A: The Time Is Now

MvG has a score of 32, and Pigfoote has a score of 6 and it is Pigfoote's turn (he has all 3 throws remaining). Feeling "generous", MvG offers Pigfoote a choice to switch his score to 35. Should he take it?

As mentioned above, according to the turn based strategy for a score of 2 to 40, 35 is the best odd score while 6 is the worst even score for Pigfoote.

According to the chart, Pigfoote will average **0.34** turns fewer to go out starting with a score of 35 (3.09 turns) vs 6 (3.43 turns). This doesn't tell the full story, however:





The charts show that for Pigfoote, a start of 35 beats the 6 despite having a lower chance to finish in 1 turn by having a higher chance of finishing in 2 or 3 turns and a lower chance of finishing in 5+ turns.

In this situation, MvG is sitting on the best score possible, and with a career checkout rate of 45% *per throw*, he has about an 83% chance to go out on his next turn. Thus the 35's big advantage in 2 and 3 turn finishes can be heavily discounted, as Pigfoote will usually not get a 2nd chance:

Wins in X Turns	**6**	**35**
1	31.4%	27.6%
2+	3.9%	5.4%
Total	**35.3%**	**33.0%**

Pigfoote should avoid falling for MvG's trap and stick with a score of 6 and expect to win about **35.3%** of the time.

Scenario B - Be Patient

This time, MvG has a score of 159, and Pigfoote a score of 35 at the start of his turn. MvG again offers Pigfoote the chance to swap scores, this time from the 35 to the 6. Should Pigfoote take it?

Tricky MvG, trying once again to outsmart the mighty Pigfoote! Not even MvG can checkout with a score of 159, which is the highest score a 3 dart checkout is not possible. Pigfoote will always get another turn if he doesn't win on this one, negating the one advantage a score of 6 has over a score of 35.

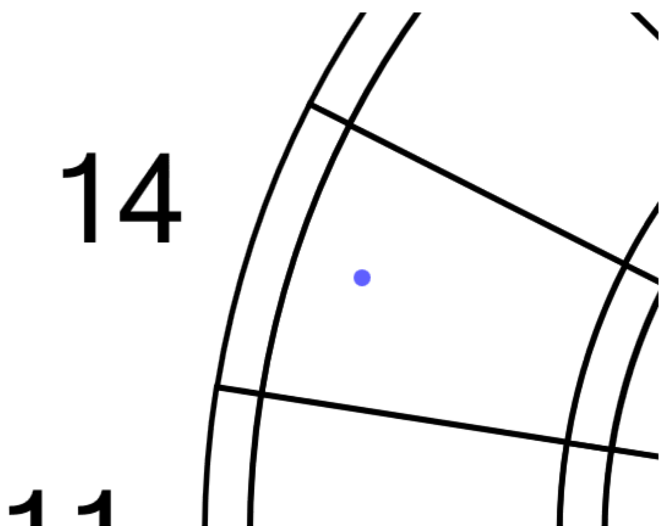
Stick with the 35 and expect to win more often than not, Pigfoote.

Scenario C - Go for Style Points

In a somewhat more realistic scenario, MvG again has 32 points, with Pigfoote at 124. While things are looking grim for the foote, he calmly nails a triple 20 on his first throw to bring his score to 64. What should he do now?

The tables above show that with a score of 64 at the start of his turn, Pigfoote should aim for a Triple 16 following either the turn based or win now strategies.

On the 2nd throw, the turn strategy still recommends the T16, but the win now strategy deviates significantly:



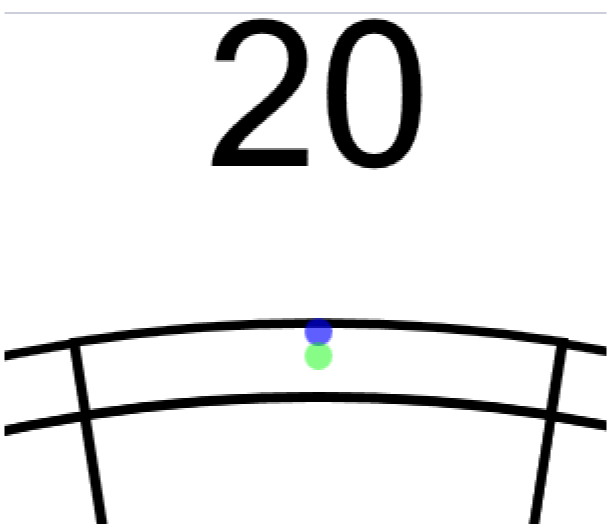
By shooting for a single 14 and planning to follow up with a throw at the double bull, Pigfoote increases his chances to go out from 2.7% to about 5.9%, a chance he should take as he likely won't get another one to win! This includes the chance that he has a "lucky" miss and instead hits a D14 on his 1st throw - setting up a much higher percentage throw at the D18 for the win.

Of course, he will lose style points for not ending with a double bull if the latter scenario occurs.

Scenario D: Meet in the Middle

MvG has 90 points. Pigfoote is up again, and he has a score of 40. Where should he aim?

While both the turn and win now strategy recommend aiming for the D20 with a score of 40, the precise locations are slightly different:

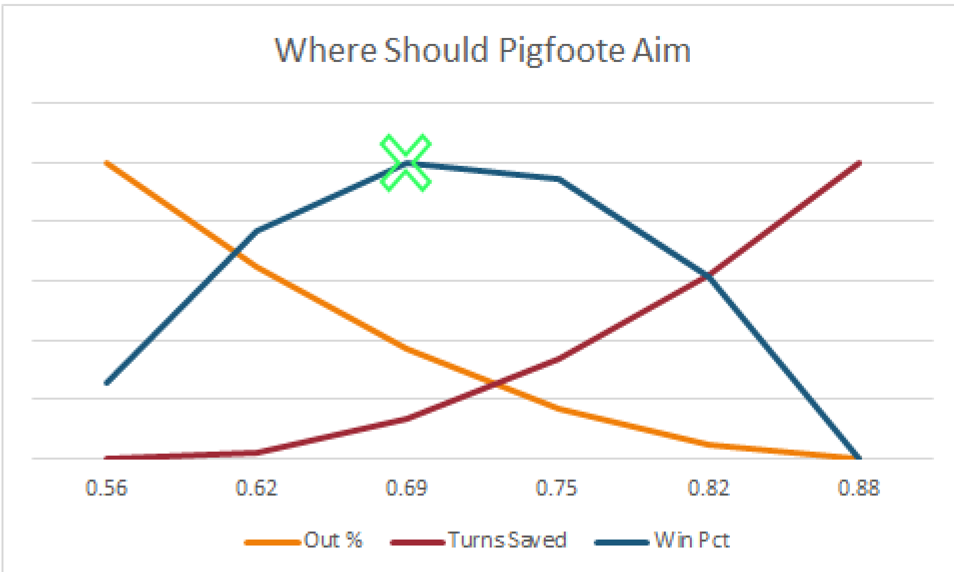


The turn optimization strategy aims towards the outer edge of the D20. It cares relatively more about staying on a good double, and is willing to give up a bit of a chance to go out to have a good score on the next turn (which the win now strategy doesn't care about, though it does care about the 2nd and 3rd throws of *this turn*).

So which one should Pigfoote aim at? The answer is somewhere in between. At a score of 90, it is about a 50/50 proposition that MvG goes out his next turn - high enough that Pigfoote should try to win now, but not high enough that he shouldn't care about his chances to go out on his next turn.

Although Pigfoote's throwing accuracy (compared to MvG's) leaves something to be desired, his targeting is precise to 0.5 mm. As the two points (which have a V of 0.56 and 0.88, respectively) are 2.5 mm away from each other, this leaves 4 points in between that he can potentially aim at. Thus, including the above target points, there are a total of 6 points to test.

The chart shows the relative* gains in checkout percentage (orange line), turns saved (red line), and expected win percentage (blue line) of all 6 targets.



*- the actual gains are very small numbers, which is why you don't see them on the y axis.

Pigfoote should aim about 1 mm outside of the green dot, which is also 1.5 mm inside the blue dot. He can expect to win about **52.2%** of the time, with his precise targeting skills accounting for a whopping **0.05%** increase in win expectancy.

If Pigfoote doesn't go out, where should MvG aim?

The score of 90 was picked for a reason - MvG's expected checkout percentage is at least 5 points higher for any score below 90, and at least 5 points less for any score above 90.

He should aim at the double bull. If he hits, he's about 70% to go out on that turn. The more likely scenario is that he hits a single bull and has a score of 65, which is another tricky score, where two strategies make sense:

- Go for a *single* bull. This has the highest chance of landing on a good double (40) for checkout, but it's a bit of an awkward target as he does *not* want to hit a double bull.
- Go for a T15. Ideally, he hits the T15 and is left with a D10 to win. A single 15 leaves him a throw at the double bull to checkout.

My charts recommend MvG follows the latter strategy - but for the 2nd throw of a turn only. On his 1st and 3rd throws, he should go for a single bull instead.

Scenario E - The High Scores Matter, Too

It's the start of the game (which means, both players have a score of 501) and Pigfoote is first. Where should he aim?

Both of Pigfoote's strategy tables recommend his optimal scoring target: the T19 (0.58,0.81). Considering MvG averages less than half as many turns to go out as Pigfoote, it is reasonable to assume that Pigfoote might want to be more aggressive than his default strategy recommends.

To test this theory, I first came up with two ranges of more aggressive targets to test:

1. Targets within the T19, but closer to the center
    a. This covers an H range of 0.5 to 0.58 and a V range of 0.5 to 0.81
2. Targets within the T20, ranging from his best scoring target (H:0.44, V:0.94) to the center.

The targets were spaced out according to Pigfoote's target precision of 0.5mm (about 0.01 H steps and 0.06 V steps, because the triple is much wider than it is tall). .

For each target, I went through Pigfoote's strategy table and for scores above 170, replaced all occurrences of the T19 with this target. For the T20 targets only, I also replaced the occurrences of the T20 in the strategy table with the target.

I then rebuilt the turn distribution tables using the updated targets, and calculated the expected win percentage against MvG's estimated turn distribution tables.

First, the T19 targets:

	Target	Win Pct
Original	T19 (0.58, 0.81)	1.051%
Center	T19 (0.5, 0.5)	1.040%
Optimized	T19 (0.56, 0.63)	1.055%

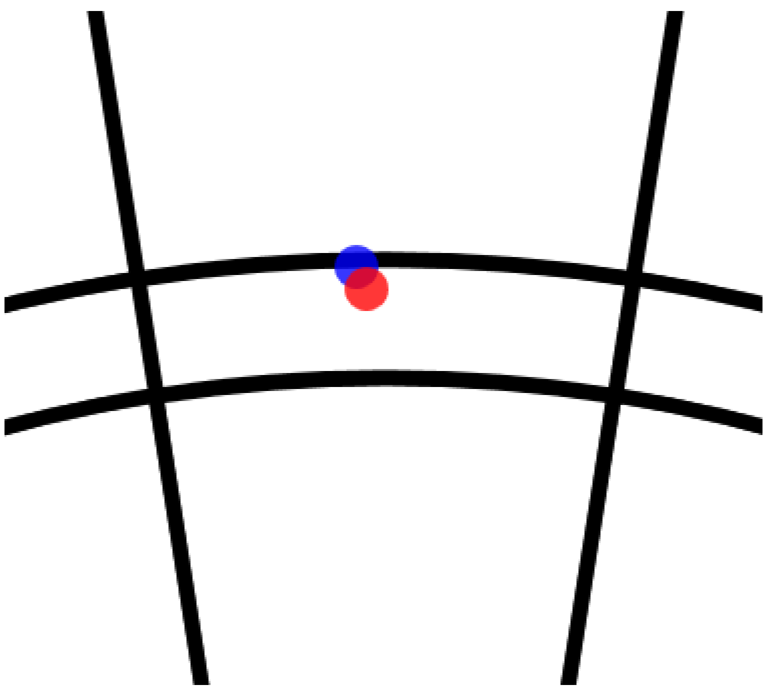
Pigfoote gains a massive 0.004% in win expectancy by tweaking his target - but he should be careful not to get too aggressive and aim at the center of the T19.

While I could argue that the relative increase of 0.4% in win expectancy is significant, I won't.

Now, the T20 targets:

	Target	Win Pct
T19 (Optimized)	T19 (0.56,0.63)	1.055%
T20 (Original)	T 20 (0.44,0.94)	1.189%
T20 (Center)	T 20 (0.50,0.50)	1.177%
T20 (Optimized)	T 20 (0.46,0.75)	**1.202%**

This time, I will argue that the relative gain of a 14% chance to win from Pigfoote's original target is significant. Pigfoote also gains almost 3 times as much by tweaking his target in the T20 compared to the T19, which is interesting because the target shifts are identical (1.5 mm vertically and 1 mm horizontally).



The red dot is where Pigfoote should aim, while the blue dot is Pigfoote's optimal scoring target for the T20.

Here is Pigfoote's turns distribution table, where Pf1 is based on Pigfoote's original strategy and Pf2 is based on Pigfoote's updated strategy.

Turns	MvG	Pf1	Pf2
3	0.08%	0.00%	0.00%
4	17.96%	0.00%	0.00%
5	45.98%	0.07%	0.10%
6	27.91%	0.94%	1.15%
7	6.43%	4.75%	5.09%
8	1.25%	12.10%	12.04%
9	0.28%	18.27%	17.66%
10+	0.10%	63.86%	63.96%

Note that since Pigfoote throws first, a tie in turns goes to Pigfoote.

As shown, Pigfoote significantly increases his chances to go out in 5, 6, and 7 turns by switching targets, which is virtually all that matters since MvG will go past 7 turns only 1.6% of the time.

I find this to be the most compelling of the five scenarios, as the gain is significant and the scenario occurs every game (of the others, only scenario C is somewhat likely to occur). Pigfoote can undoubtedly improve upon the 1.2% chance to win further down the line by going through a similar process for each subsequent throw.

Finishing the 501 Solution

In this section I will first address the bust rule used most commonly in competitive darts, and then attempt to solve the complete win maximization strategy for Pigfoote (which of course could be done for any player that we have a pdf for).

Addressing the More Common Bust Rule

If a bust results in the players score being returned to the value at the beginning of the turn, this can alter the strategy for the 2nd and 3rd throws of a turn significantly.

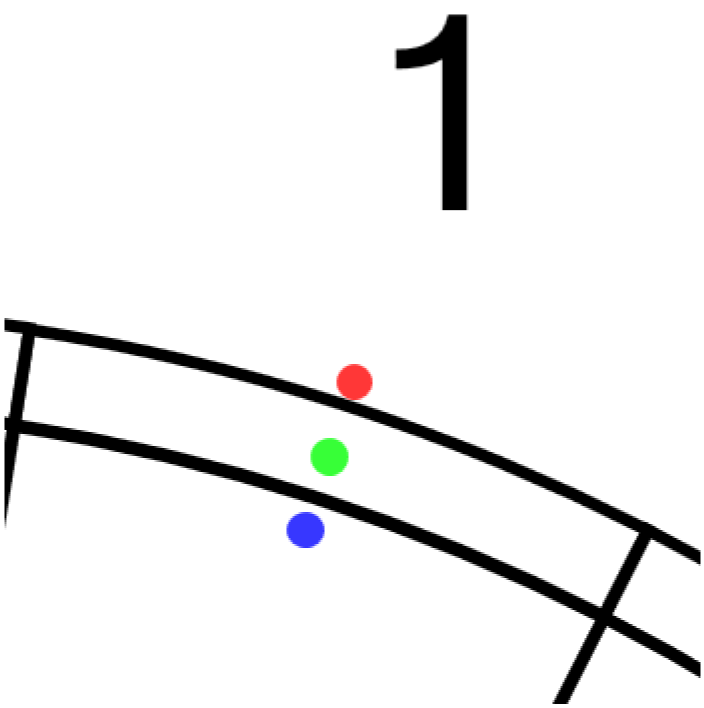
Consider the following scenarios where Pigfoote has a score of 2 on the last throw of his turn:

A. He started the turn with a score of 3, and threw a 1 and then missed the board.
B. He started with a score of 8, and threw a 4 and then a 2.

With the alternate bust rule, Pigfoote should go for a D1 and not care about anything else because the result is the same for anything but the D1. With this bust rule, we have to consider that his next turn might start with a score of either 2 or 3 for scenario A, and either 2 or 8 for scenario B.   Here are Pigfoote's values for these scores with the alternate bust rule:

Score	Out	Turns
2	29.6%	3.38
3	17.7%	3.93
8	37.9%	2.90

To find out where Pigfoote should aim in these scenarios, we can plug in the values of the 3 and the 8 from this table as an approximation. Here are the results:



	Target	Turns
Alt Bust	D 1 (0.50,0.44)	3.83
A	M 1 (0.50,0.04)	4.02
B	OS 1 (0.50,0.95)	3.59

In both scenarios, Pigfoote saves about 0.04 expected turns by moving his target away from the green dot. Of course, the values I used were based on a table built based on the alternate bust rule and are not precise. The same iterative approach used to solve for scores of 3 plus to build the original tables can be used here to arrive at a definitive answer for Pigfoote. On the 2nd pass the targets might grow further apart as this bust rule makes a score of 8 even more favorable compared to a score of 3.

Unfortunately, the size of Pigfoote's strategy tables will have to increase substantially to account for all score/throw/starting score combinations where a bust is possible on that turn. The full table needs 111 more entries (the number of scores possible in 2 throws excluding zero) for the last throw of each score from 2 to 61, and 42 more entries for the 2nd throw of each score from 2 to 121. This increases Pigfoote's strategy table from 1500 to 13200 entries.

Note that this bust rule has by far the greatest impact on Pigfoote's strategy specifically with a score of 2, since this is the only score where a single of the target he is aiming at results in a bust. The difference in the rules does not alter any of Pigfoote's recommended strategies in his match against MvG, for example.

A Win Optimization Strategy Table

Pigfoote vs. Jones

As demonstrated in the examples above, Pigfoote's full strategy table can be converted into turn distribution tables for each entry in the table.

Suppose another player (I'll call him *Jones*) of exactly the same skill has an upcoming match against Pigfoote. In this match, the players will play a googolplex games and alternate throwing first. Jones found out that Pigfoote will exactly follow his turn-minimization strategy tables, . and used this to build his own tables optimized to win specifically against Pigfoote.

Jones' *complete* tables require 500 times as many entries as Pigfoote's but are built using the same process. The tricky part is that to build his own tables, Jones needs to consider the range of Pigfoote's possible scores on his next turn to evaluate the relative value of each of his own scores.

For example, suppose Jones has a score of 35 on the last throw of his turn while Pigfoote has a score of 32. I will use the notation (35-3,32) - where 3 is the throw number of the turn to represent this scenario. To find his optimal target, Jones needs to convert Pigfoote's 32 into the following table:

End of Turn	Pct
Out	40.01%
16	20.44%
8	14.67%
32	14.65%
4	3.76%
24	2.17%
12	1.28%
18	0.64%
4	0.56%
25	0.48%
6	0.32%
22	0.31%
9	0.24%
Others*	0.49%

*-of course, Jones will not lump the other scores all together and calculate each one individually, no matter how rare an occurrence.

Jones uses this table along with previous entries in his own table to create a single value (expected winning percentage) for each of his possible end of turn scores based on the weighted product of Pigfoote's next turn scores. For example, the value of Jones hitting a 3 (leaving his score at 32 for his next turn) is: 0.4001 * 0 (he loses if Pigfoote goes out his next turn) + 0.2044 * (32-1,16) + 0.1467 * (32-1,8) + 0.1465 * (32-1,32) and so on.

For a computer, this additional step is a fast, but non-trivial calculation that unfortunately needs to be done quite a lot as Jones' full table will contain 6.6 million entries.

But as a result of his hard work, Jones ends up winning this match against Pigfoote.

The Rematch

Satisfied with his results, an over-confident Jones accepts a rematch, and then takes an extended vacation without revising his strategy. A secret operative within Pigfoote's organization discovers Jones' 6.6 million record strategy table. Pigfoote uses this to create a strategy optimized to beat Jones. Pigfoote wins the rematch.

The Final Battle

The two players agree to play a 3rd and final deciding match. To assure that the winner is truly deserving, this match will not end after a googolplex games, but rather when one player has won a googolplex games *more* than the other player (they are not concerned about time, as they are able to play a googolplex games per nanosecond).

While Jones is out for vengeance, Pigfoote also does not rest after his win. Both players assume that the other player will try to outsmart the other by thinking one step ahead. They arrive at exactly the same strategy by the following process:

Starting with Pigfoote's 2nd match strategy, a strategy is built to defeat this. Then a strategy is built to defeat the strategy just built, and so on. After some number of iterations, the entire 6.6 million strategy table will be completely unchanged. The resulting strategy *cannot be beat* (since the table was unchanged when we tried to beat it), though, it can be tied by following the same strategy.

Sadly, neither Pigfoote nor Jones has been seen since the match (which took place in an undisclosed location) began.

Optimal Strategies for The Game of Darts (1982 Research Paper)

The author takes a practical approach to creating target recommendations for the x01 variant.

The practical simplifications are fairly reasonable, particularly considering the hardware available in 1982:

- Instead of considering all possible target points, it considers 24 target points for each number section.
- It focuses on turn minimization only and does not address win expectancy (though it mentions the one obvious case - a score of 50 against a strong player - where the best strategy might be different than recommended).
- For scores of above *107*, it recommends a general strategy of aiming at the optimal scoring target.
- It groups players into 5 categories of skill with a range of σ values in each category.
- It considers the (easier to solve) alternate bust rule only.

The paper is available here, but not for free download. I ended up purchasing a 1 month subscription to JSTOR for $19.50 (thankfully, unlike most websites you can actually do this with an option *not* to auto-renew your subscription).

I was generally impressed with this paper (especially given it was 20 years ahead of the other works cited above) and found no differences in it's recommendations that are not explainable by the above simplifications. It would have been a challenge for me to do all of this work in 1982, partly because of the hardware limitations, but mainly because I was only a year old.

Solved?

The vast majority of my work for Part 1 was done almost 3 years ago (2016), though I am only now writing about it.

When I recently decided to resume the project, my original plan was to finish the work described and write a paper on it, which I planned on naming "Solving Darts."

The problem is - I didn't actually solve darts. What I solved is a computer simulation of darts, all without owning a dartboard.

One of the reasons that I came close to, but didn't quite finish all of this work 3 years ago is that all of the math is dependant on a few assumptions that I question apply to human dart players:

1. That a player's throwing distribution follows a bivariate normal distribution, though none of the work depends on this and would work with any probabilistic density function.
2. That a player's throws are independent of one another (ie, perhaps "hot" and "cold" are things that matter and not explained by randomness)
3. That a player throws exactly the same at every target on the dartboard.
4. That if a player switches targets on his dartboard and throws a lot of darts at this new target, he will throw exactly the same for his first throws as he does his last throws.
5. That a player can aim at specific points to the precision suggested by my work, the geek, and the statistician playing darts.

6. That the average player should aim at a target that produces a low score (the inner portion of the single 7). Admittedly, that is not an argument against the math but rather the main objective of the game of darts - to have fun.

Which leads to Part 2…